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Semilinear elastic waves with different damping mechanisms

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Weitere Personen waren an der Abfassung der vorliegenden Arbeit nicht beteiligt.

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Declaration

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July 2020 B. Sc. Chen, Wenhui

Contents

1.Introduction

1.1. Background

Elastic waves describe particles vibrating in materials holding the property of elasticity. Moreover, when the particles are moving and there exists a force acts on the particles restore them to their original position, elastic waves will be produced. Elastic waves have been widely applied in describing wave propagation in an elastic medium, such as seismic waves in Earth and ultrasonic waves in human body.

The theory of elastic waves was established by [74], where the author gave an equation of motion for the displacement of a particle in elastic solids. Next, Cauchy developed the mathematical theory of the elasticity by introducing the notions of stress, strain and stress-strain relations in [6]. Also, Poisson in [83], solved the differential equation of motion for an elastic solid by decomposing the displacement into an irrotational and a circular part, each part being a solution of a wave equation. In the reference [33], Green derived the equation of motion and established the boundary conditions at the surface of an elastic solid by applying energy and variational principles.

For the sake of briefness, we derive the equations governing motion of an infinite, isotropic and homogeneous elastic continuum, which can be described by

$$u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u = 0, \tag{1.1}$$

where the positive constants a^2 and b^2 are related to the Lamé constants satisfying $b > a > 0$. The system of elastic waves satisfies the property of finite speed of propagation given by the coefficient b, which is the speed of propagation of the longitudinal P-wave. The coefficient a is the speed of propagation of the transverse S-wave. These properties have been discussed in the paper [7].

In general, one can not expect that the system (1.1) models real-world problems, because there exist several kinds resistance, such as fluid resistance and frictional resistance. We always use damping mechanisms to describe oscillation amplitude, which are reduced through the irreversible removal of the vibratory energy in a mechanical system or a component (see [34]). In other words, the damping mechanisms can be considered as a loss of energy. As we will see later, some damping mechanisms in the elastic waves (1.1) can be described by

$$u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + Au_t = 0, \tag{1.2}$$

where the term Au_t describes different kinds of damping mechanisms. In fact, different type damping mechanisms will make different influences on the type of losing energy. It means the type of dissipation affects the decay rate and the region of energy decay. What's more, due to the structure of the damping mechanisms, some properties of elastic displacements are different.

However, in various physical and chemical problems, as well as their abstract form in applied mathematics, we always encounter semilinear problems rather than linear problems (1.1) and (1.2). For example, semilinear elastic waves can be modeled by

$$u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + Au_t = f(u), \tag{1.3}$$

where A is allowed to be the zero operator. Here $f(u)$ is a nonlinear term, which will be introduced later.

1.2. A review of some previous results

In this section, let us start recalling briefly some results for wave equations and elastic waves with or without damping mechanisms.

1.2.1. Cauchy problem for elastic waves

Let us begin with the consideration of linear elastic waves. For linear elastic waves in 3D

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u = 0, & x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $b > 2a/\sqrt{3} > 0$, by using the Helmholtz decomposition the authors of [3] derived Strichartz estimates of solutions to (1.4). Moreover, taking $b > a > 0$ in the Cauchy problem (1.4) and supposing that $u \in \big(\mathcal{C}^\infty\big([0, T], \mathcal{C}_0^\infty\big(\mathbb{R}^3\big)\big)\big)^3$, [36] proved Keel-Smith-Sogge type estimates of solutions to (1.4) by using the Hodge decomposition together with the multiplier method. In the book [85], the author showed the proof of $L^p - L^q$ estimates on the conjugate line for some energies by using the interpolation between corresponding $L^2 - L^2$ and $L^1 - L^\infty$ estimates. For the weighted estimates with the angular momentum operators (generator of the spatial rotation) to the inhomogeneous linear elastic waves, we refer to [53, 57, 97, 56, 1, 98, 114, 36]. In general, they applied the method of vector fields with suitable weighted functions.

Let us formally consider $a = b = 1$ in linear elastic waves (1.4). Then, the system of elastic waves can be reduced to free wave equation as follows:

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.5}$$

It is well-known that the representation of solutions is d'Alembert's formula in one space dimension, Poisson's formula in two space dimensions, Kirchhoff's formula in three space dimensions. For representations in higher dimensions, we refer to the book [26]. Estimates of solutions to the free wave equation (1.5) have been investigated by many mathematicians, for example, $L^p - L^q$ estimates of solutions have been derived in [63, 102, 103, 5, 82, 85]. Especially, $L^p - L^p$ estimates have been developed in [82], whose method is basing on the boundedness of related Fourier integral operators.

1.2.2. Cauchy problem for damped elastic waves

Let us turn to elastic waves with different damping mechanisms. In recent years, there are some papers devoted to the study of linear elastic waves with friction ($\theta = 0$) or structural damping ($\theta \in (0, 1]$), which are modeled by

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + (-\Delta)^\theta u_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

where $b > a > 0$. In the paper [41], the authors considered the Cauchy problem (1.6) with $\theta \in [0, 1]$ and initial data is taken from energy spaces with additional L^1 regularity to obtain almost sharp energy estimates by using energy methods in the Fourier space.

For the Cauchy problem (1.6) with $\theta \in (0, 1]$, the author of [88] studied qualitative properties of solutions, including Gevrey smoothing if $\theta \in (0, 1)$, propagation of singularities if $\theta = 1$, and estimates of higher-order energies.

More general structural damping mechanisms in elastic waves are considered in [109]. Especially, the authors studied energy estimates for the following linear elastic waves with Kelvin-Voigt damping:

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + \mathbb{E}u_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.7}$$

with the Lamé operator $\mathbb{E} := -a^2\Delta - \big(b^2 - a^2\big)\nabla \operatorname{div}$ for $n \geqslant 2$. By applying the Haraux-Komornik inequality and energy methods in the Fourier space, the authors proved almost sharp energy estimates.

Let us formally consider $a = b = 1$ in linear damped elastic waves (1.6). The system of damped elastic waves can be immediately reduced to the following wave equation with friction or structural damping:

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^\theta u_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.8}$$

Let us consider the wave equation with friction (i.e., $\theta = 0$ in (1.8)) in the first place. Sharp decay estimates of solutions to (1.8) with $\theta = 0$ was initially derived by [67]. Moreover, it is well-known that the solution of (1.8) with $\theta = 0$ has the effect of *diffusion phenomenon*, which means that the solution to (1.8) with $\theta = 0$ behaves like the solution of the heat equation as $t \to \infty$ as follows:

$$\begin{cases} v_t - \Delta v = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ v(0, x) = v_0(x) = u_0(x) + u_1(x) & x \in \mathbb{R}^n. \end{cases} \tag{1.9}$$

This phenomenon has been well-studied in [37, 38, 66, 72, 75, 76, 77, 112].
Next, we take structural damping in damped wave equation (i.e., $\theta \in (0, 1)$ in (1.8)). In recent years, qualitative properties of solutions to (1.8) with $\theta \in (0, 1)$ have been widely investigated. We refer to [29] for well-posedness and smoothing effect of solutions, [24, 25, 22] for estimates of solutions, [21, 111] for diffusion phenomena. Particularly, for (1.8) with $\theta \in (0, 1/2)$, [21] showed *double diffusion phenomena* such that the solution u behaves like the solution v^+ to

$$\begin{cases} v_t^+ + (-\Delta)^{1-\theta} v^+ = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ v^+(0, x) = v_0^+(x) & x \in \mathbb{R}^n, \end{cases}$$

and the difference $u - v^+$ behaves like the solution to

$$\begin{cases} v_t^- + (-\Delta)^{\theta} v^- = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ v^-(0, x) = v_0^-(x) & x \in \mathbb{R}^n, \end{cases}$$

in the $L^p - L^q$ framework, where $1 \leqslant p \leqslant q \leqslant \infty$. Here $v_0^+ = v_0^+(x)$ and $v_0^- = v_0^-(x)$ are suitably choice of initial data, which will be defined by the aid of $u_0 = u_0(x)$ as well as $u_1 = u_1(x)$.
When we consider $\theta = 1$ in the Cauchy problem (1.8), we interpret the model by wave equation with viscoelastic damping or strong damping. The well-posedness of solutions has been studied in [47, 29, 25]. Some estimates of solutions have been developed in [25, 24], too. Furthermore, $L^p - L^q$ estimates of solutions have been investigated in [84, 94]. Lastly, we mention that the asymptotic profiles of solutions with a framework of weighted L^1 data, has been studied in [40, 69, 70].

Next, we consider the following semilinear damped wave models:

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^\theta u_t = |u|^p, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.10}$$

with $\theta \in [0, 1]$. The global (in time) existence of solutions and blow-up of solutions have been investigated in recent years, which can be summarized as follows:

- In the case when $\theta = 0$: the critical exponent is *Fujita exponent* $p_{\text{Fuj}}(n) = 1 + \frac{2}{n}$ (one can see [106, 46, 115]);
- In the case when $\theta \in (0, 1/2)$: the global (in time) unique solutions exist if $p > 1 + \frac{2}{n-2\theta}$ (one can see [20, 24]);
- In the case when $\theta = 1/2$: the critical exponent is $p_{\text{crit}}(n) = 1 + \frac{2}{n-1}$ (one can see [18, 20, 24]);
- In the case when $\theta \in (1/2, 1]$: the global (in time) unique solutions exist if $p > 1 + \frac{1+2\theta}{n-1}$ (one can see [24]).

Lastly, let us turn to weakly coupled systems with $p, q > 1$, for example, to

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^\theta u_t = |v|^p, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ v_{tt} - \Delta v + (-\Delta)^\theta v_t = |u|^q, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.11}$$

where $\theta = 0$ or $\theta = 1/2$. Critical exponents to the system (1.11) with $\theta = 0$ are described by the condition

$$\alpha_{\max} = \max\Big\{\frac{p+1}{pq-1}; \frac{q+1}{pq-1}\Big\} = \frac{n}{2}.$$

In [104] the authors investigated for $n = 1, 3$, that if $\alpha_{\max} < \frac{n}{2}$, then there exists a unique global (in time) Sobolev solution for small data. If $\alpha_{\max} \geqslant \frac{n}{2}$, then every local (in time) solution, in general, blows up in finite time. The paper [73] generalized their existence results to $n = 1, 2, 3$ and improved the time decay estimates when $n = 3$. The recent paper [78] determined the critical exponents for any space dimension n. The proof of the global (in time) existence of energy solutions is based on the weighted energy method. Later [105] considered the following generalization of the model (1.11) with $\theta = 0$:

$$\begin{cases} u^1_{tt} - \Delta u^1 + u^1_t = |u^k|^{p_1}, \\ u^2_{tt} - \Delta u^2 + u^2_t = |u^1|^{p_2}, \\ \quad \vdots \\ u^k_{tt} - \Delta u^k + u^k_t = |u^{k-1}|^{p_k}, \end{cases} \tag{1.12}$$

where $k \geqslant 2$ and $p_j > 1$ for $j = 1, \ldots, k$. We define the matrix P as

$$P = \begin{pmatrix} 0 & 0 & \cdots & 0 & p_1 \\ p_2 & 0 & \cdots & 0 & 0 \\ 0 & p_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_k & 0 \end{pmatrix}$$

and consider $P - I_{k\times k}$. Therefore, it is clear that

$$|P - I_{k\times k}| = (-1)^{k-1}\Big(\prod_{j=1}^{k} p_j - 1\Big),$$

and the inverse matrix of $P - I_{k\times k}$ exists because $|P - I_{k\times k}| \neq 0$. Then, we can define

$$\alpha = (\alpha_1, \ldots, \alpha_k) = (P - I_{k\times k})^{-1} \cdot (1, \ldots, 1)^{\mathrm{T}}.$$

The author of [105] proved that when $n \leqslant 3$, then the critical exponents of the system (1.12) are described by the condition

$$\alpha_{\max} = \max\{\alpha_1; \ldots; \alpha_k\} = \frac{n}{2}.$$

In addition, the author obtained blow-up results for any space dimensions. Then, the paper [79] determined the critical exponents for any space dimension n, where the weighted energy method has been used in the proof of the global (in time) existence of solutions.

For $\theta = 1/2$ in (1.11), the paper [19] has shown that the global (in time) existence of small data Sobolev solution holds if

$$\alpha_{\max} = \max\Big\{\frac{p+1}{pq-1}; \frac{q+1}{pq-1}\Big\} < \frac{n-1}{2},$$

and $n \geqslant 2$. On the contrary, the nonexistence result for global (in time) solutions holds if

$$\alpha_{\max} > \frac{n-1}{2}$$

holds for $n \geqslant 1$.

1.3. Objectives and structure of the thesis

In the present thesis we are going to consider the Cauchy problem for linear elastic waves

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + Au_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.13}$$

where $b > a > 0$, $u = \big(u^1, \dots, u^n\big)^{\mathrm{T}}$ with $n = 2, 3$, and the corresponding semilinear elastic waves with power source nonlinearities, that is,

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + Au_t = f(u), & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.14}$$

Here the operator

$$A :\ u_t \in \mathbb{R}^n \to Au_t \in \mathbb{R}^n \quad \text{for} \quad n = 2, 3,$$

has several forms in the thesis. Specifically, we will discuss the following operators A, which exert different influences on the properties of solutions to the Cauchy problems (1.13) and (1.14):

- elastic waves without any damping, that is, $Au_t = 0$;
- elastic waves with *friction*, that is, $Au_t = u_t$;
- elastic waves with *structural damping*, that is, $Au_t = (-\Delta)^\theta u_t$ with $\theta \in (0, 1]$, especially, $Au_t = -\Delta u_t$ denotes *viscoelastic damping*;
- elastic waves with *Kelvin-Voigt damping*, that is, $Au_t = \big(- a^2\Delta - \big(b^2 - a^2\big)\nabla \operatorname{div} \big)u_t$ with $b > a > 0$.

Moreover, the power source nonlinearities on the right-hand sides of (1.14) have the following forms:

$$\begin{aligned} f(u) &= \big(|u^2|^{p_1}, |u^1|^{p_2}\big)^{\mathrm{T}} && \text{when} \quad x \in \mathbb{R}^2, \\ f(u) &= \big(|u^3|^{p_1}, |u^1|^{p_2}, |u^2|^{p_3}\big)^{\mathrm{T}} && \text{when} \quad x \in \mathbb{R}^3, \end{aligned}$$

where $p_1, p_2, p_3 > 1$.

Let us sketch the division of the thesis in chapters, and the topic therein developed.

In Chapter 2 some qualitative properties of solutions to linear elastic waves with friction or structural damping in 3D are derived, including smoothing effect, propagation of singularities, L^2 well-posedness, energy estimates with initial data taking from different function spaces and diffusion phenomena. In order to derive these properties, firstly representations of solutions are obtained by using WKB analysis and diagonalization procedure.

In Chapter 3 several global (in time) existence results are proved for weakly coupled systems of semilinear elastic waves with friction or structural damping in 3D. The results of global (in time) existence of energy solutions, Sobolev solutions with higher and large regularities are proved.

In Chapter 4 by using asymptotic expansions of eigenvalues and corresponding eigenprojections, the results of energy estimates with or without additional regularity and diffusion phenomena for linear elastic waves with Kelvin-Voigt damping in 2D are obtained. For three dimensional case, it allows us to apply the Helmholtz decomposition. Moreover, some results for global (in time) existence of small data energy solutions to weakly coupled systems of semilinear elastic waves with Kelvin-Voigt damping in 2D and 3D are proved.

In Chapter 5 by using explicit representation of the solution to three dimensional linear elastic waves without any damping, the results of H^s well-posedness and $L^p - L^q$ estimates not necessary on the conjugate line are derived, particularly, $L^p - L^p$ estimates with $p \in (1, \infty)$. Then, some estimates for radial solutions are obtained. Finally, we give a conjecture on an open problem of semilinear elastic waves in 3D.

In Chapter 6 some results for other topics related to coupled systems, which have been studied in the Ph.D. period of the author of the thesis, are shown. First of all, global (in time) existence of small data solutions and blow-up of solutions to the weakly coupled system of semilinear wave equations with distinct scale-invariant terms in the linear part are shown. Secondly, qualitative properties of solutions to linear thermoelastic plate equations with friction or structural damping are shown. Thirdly, a new threshold of diffusion phenomena for doubly dissipative elastic waves in 2D is shown.

In Chapter 7 some concluding remarks and further researches complete the thesis.

Finally, we mention that some of the results derived in the present thesis are already published (c.f. [17, 16, 10, 9, 8]), or under review (c.f. [64, 14, 12, 15, 13, 11]).

2. Linear elastic waves with friction or structural damping in 3D

2.1. Introduction

In this chapter we are interested in the study of the following Cauchy problem for linear elastic waves with friction or structural damping in 3D:

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + (-\Delta)^\theta u_t = 0, & x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^3, \end{cases} \tag{2.1}$$

with $b > a > 0$ and $\theta \in [0, 1]$, where $u = \big(u^1, u^2, u^3\big)^{\mathrm{T}}$. Here $\theta = 0$ appears in the model with *friction* or *external damping*, $\theta \in (0, 1]$ appears in the model with *structural damping*, in particular, $\theta = 1$ appears in the model with *viscoelastic damping* or *strong damping*.

It is well-known that different damping mechanisms have different influences on qualitative properties of solutions. Therefore, in this chapter we focus on qualitative properties of solutions to the Cauchy problem (2.1) influenced by the parameter $\theta \in [0, 1]$ in the damping term. Our main purpose is to study smoothing effect, propagation of singularities, energy estimates with different data spaces and diffusion phenomena. To derive these properties of solutions, we obtain representations of solutions in the first step. However, because the operator $(-\Delta)^\theta$ with $\theta \in [0, 1]$ acts on u_t in the damping term, we may not directly apply the spectral theory associated with asymptotic expansions of eigenvalues and their corresponding eigenprojections in coupled systems (see, for example, [39, 86, 10]). To overcome this difficulty, we may derive representations of solutions by applying WKB analysis. The main tool is the application of a multi-step diagonalization procedure, which was mainly proposed in [90, 49, 88]. On the one hand, concerning energy estimates for solutions of the Cauchy problem (2.1), the recent paper [41] derived almost sharp energy estimates by employing energy methods in the Fourier space. But, sharp energy estimates for the Cauchy problem (2.1) are not clear. On the other hand, diffusion phenomena for wave equations with friction (see [54, 66, 37, 72, 77, 68]) or structural damping (see [21, 40, 70, 69, 45]) have been well-studied. Nevertheless, concerning diffusion phenomena for elastic waves with friction or structural damping, it seems that we still do not have any previous research manuscripts. In this chapter we give the answer in the three-dimensional case.

The rest of the chapter is organized as follows. In Section 2.2 by dividing frequencies of the phase space into different zones and employing diagonalization procedure in each zone, we derive representations of solutions. Then, basing on these representations of solutions we study qualitative properties of solutions to the Cauchy problem (2.1). More specifically, in Section 2.3 we investigate the smoothing effect and propagation of singularities of solutions. In Section 2.4 we obtain energy estimates with different regularity initial data. Next, by introducing a series of reference systems, we derive diffusion phenomena. In the last section the summary of this chapter, and some concluding remarks complete this chapter.

2.2. Asymptotic behavior of solutions

Let us begin with rewriting the system in the Cauchy problem (2.1) in the following form:

$$u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\begin{pmatrix} \partial_{x_1}^2 & \partial_{x_1x_2} & \partial_{x_1x_3} \\ \partial_{x_2x_1} & \partial_{x_2}^2 & \partial_{x_2x_3} \\ \partial_{x_3x_1} & \partial_{x_3x_2} & \partial_{x_3}^2 \end{pmatrix} u + (-\Delta)^\theta u_t = 0. \tag{2.2}$$

Constructing the corresponding ordinary differential system by applying the partial Fourier transformation with respect to spatial variables to (2.2), we obtain that $\hat{u}(t,\xi) = \mathcal{F}_{x\to\xi}(u(t,x))$ solves

$$\hat{u}_{tt} + |\xi|^{2\theta}\hat{u}_t + |\xi|^2 A(\eta)\hat{u} = 0, \tag{2.3}$$

where $\eta = \xi/|\xi| \in \mathbb{S}^1$ and

$$A(\eta) = \begin{pmatrix} a^2 + \left(b^2 - a^2\right)\eta_1^2 & \left(b^2 - a^2\right)\eta_1\eta_2 & \left(b^2 - a^2\right)\eta_1\eta_3 \\ \left(b^2 - a^2\right)\eta_1\eta_2 & a^2 + \left(b^2 - a^2\right)\eta_2^2 & \left(b^2 - a^2\right)\eta_2\eta_3 \\ \left(b^2 - a^2\right)\eta_1\eta_3 & \left(b^2 - a^2\right)\eta_2\eta_3 & a^2 + \left(b^2 - a^2\right)\eta_3^2 \end{pmatrix}.$$

Our assumption $b > a > 0$ implies that the matrix $A(\eta)$ is positive definite. The eigenvalues of $A(\eta)$ are a^2, a^2 and b^2. For further approach, we introduce

$$M(\eta) := \begin{pmatrix} -\frac{\eta_2}{\eta_1} & -\frac{\eta_3}{\eta_1} & \frac{\eta_1}{\eta_3} \\ 1 & 0 & \frac{\eta_2}{\eta_3} \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A_{\text{diag}}(|\xi|) := |\xi|^2 \text{diag}\left(a^2, a^2, b^2\right).$$

By making a new variable

$$v(t,\xi) := M^{-1}(\eta)\hat{u}(t,\xi),$$

we have

$$\begin{cases} D_t^2 v - i|\xi|^{2\theta} D_t v - A_{\text{diag}}(|\xi|)v = 0, & \xi \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ (v, D_t v)(0,\xi) = (v_0, -iv_1)(\xi), & \xi \in \mathbb{R}^3, \end{cases} \tag{2.4}$$

where $v_j(\xi) = M^{-1}(\eta)\hat{u}_j(\xi)$ for $j = 0,1$. After introducing $W^{(0)} = W^{(0)}(t,\xi)$ by

$$W^{(0)}(t,\xi) := \begin{pmatrix} D_t v(t,\xi) + A_{\text{diag}}^{1/2}(|\xi|)v(t,\xi) \\ D_t v(t,\xi) - A_{\text{diag}}^{1/2}(|\xi|)v(t,\xi) \end{pmatrix}, \tag{2.5}$$

we derive the Cauchy problem for the evolution system

$$\begin{cases} D_t W^{(0)} - \frac{i}{2}|\xi|^{2\theta} B_0 W^{(0)} - |\xi| B_1 W^{(0)} = 0, & \xi \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ W^{(0)}(0,\xi) = W_0^{(0)}(\xi), & \xi \in \mathbb{R}^3, \end{cases} \tag{2.6}$$

with matrices B_0 and B_1, which are given by

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & -b \end{pmatrix}.$$

Remark 2.2.1. *According to the papers [95, 108], we may interpret the system* (2.6) *as a symmetric hyperbolic-parabolic coupled system when $\theta = 1$, and as a symmetric hyperbolic coupled system with a symmetric relaxation when $\theta = 0$. Therefore, one may derive a dissipative structure of the Cauchy problem* (2.6) *with $\theta = 0, 1$, by applying Lemma 2.2 in [108]. However, the other properties (e.g. smoothing effect, diffusion phenomena) are still not clear. Furthermore, concerning the remaining case when $\theta \in (0,1)$, the theory for dissipative structure to a general class of symmetric hyperbolic-parabolic systems in [95, 108] cannot be applied because the non-local operator $(-\Delta)^\theta$ appears.*

Before applying WKB analysis, let us define the cut-off functions $\chi_{\text{int}}, \chi_{\text{mid}}, \chi_{\text{ext}} \in \mathcal{C}^\infty(\mathbb{R}^3)$ having their supports in the following zones:

$$\begin{aligned}\mathcal{Z}_{\text{int}}(\varepsilon) &:= \left\{\xi \in \mathbb{R}^3 : |\xi| < \varepsilon\right\},\\ \mathcal{Z}_{\text{mid}}(\varepsilon) &:= \left\{\xi \in \mathbb{R}^3 : \varepsilon \leqslant |\xi| \leqslant \tfrac{1}{\varepsilon}\right\},\\ \mathcal{Z}_{\text{ext}}(\varepsilon) &:= \left\{\xi \in \mathbb{R}^3 : |\xi| > \tfrac{1}{\varepsilon}\right\},\end{aligned}$$

respectively, so that $\chi_{\text{mid}} = 1 - \chi_{\text{int}} - \chi_{\text{ext}}$. Here $\varepsilon > 0$ is a sufficiently small constant.

To understand the influence of the parameter $|\xi|$, we distinguish between the following cases.

Case 2.0: We consider $\theta \neq 1/2$ with $\xi \in \mathcal{Z}_{\text{mid}}(\varepsilon)$.
Case 2.1: We consider $\theta \in [0, 1/2)$ with $\xi \in \mathcal{Z}_{\text{int}}(\varepsilon)$ or $\theta \in (1/2, 1]$ with $\xi \in \mathcal{Z}_{\text{ext}}(\varepsilon)$.
Case 2.2: We consider $\theta \in [0, 1/2)$ with $\xi \in \mathcal{Z}_{\text{ext}}(\varepsilon)$ or $\theta \in (1/2, 1]$ with $\xi \in \mathcal{Z}_{\text{int}}(\varepsilon)$.
Case 2.3: We consider $\theta = 1/2$ for all frequencies $\xi \in \mathbb{R}^3$.

To obtain the asymptotic behavior of solutions to the Cauchy problem (2.6), the diagonalization schemes (c.f. [49], [88], [90]) in the small and large frequency zones, and energy method in the Fourier space in the middle frequency zone are available.

2.2.1. Energy method in the Fourier space for Case 2.0

Our motivation of this subsection is to prove an exponential decay result by using energy methods in the Fourier space to guarantee an exponential stability of solutions.

Theorem 2.2.1. *Let us consider frequencies in the zone $\mathcal{Z}_{\text{mid}}(\varepsilon)$. The solution $v = (v^1, v^2, v^3)^{\mathrm{T}}$ to the Cauchy problem* (2.4) *satisfies the estimate for all $t > 0$*

$$\left|v_t^k\right|^2 + |\xi|^2\left|v^k\right|^2 \lesssim \mathrm{e}^{-ct}\sum_{k=1}^{3}\left(\left|v_0^k\right|^2 + \left|v_1^k\right|^2\right),$$

where $k = 1, 2, 3$ and c is a positive constant.

Proof. Defining $\varpi_k = a^2$ for $k = 1, 2$, and $\varpi_k = b^2$ for $k = 3$, we introduce an energy $E_{\text{mid}}(v) = E_{\text{mid}}(v)(t, \xi)$ in the zone $\mathcal{Z}_{\text{mid}}(\varepsilon)$ according to the structure of (2.4) by

$$E_{\text{mid}}(v) := \frac{1}{2}\sum_{k=1}^{3}\left(\left|v_t^k\right|^2 + \varpi_k|\xi|^2\left|v^k\right|^2\right).$$

Multiplying the equation in (2.4) by the function $\bar{v}_t$ we obtain

$$\frac{\partial}{\partial t}E_{\text{mid}}(v) = -|\xi|^{2\theta}\sum_{k=1}^{3}\left|v_t^k\right|^2. \tag{2.7}$$

Homoplastically, we multiply the equation in (2.4) by the function $\bar{v}$ and take the real part of the resulting identity to arrive at

$$\frac{1}{2}\sum_{k=1}^{3}\varpi_k|\xi|^2\left|v^k\right|^2 + \frac{\partial}{\partial t}\Big(\sum_{k=1}^{3}\mathrm{Re}\left(\bar{v}^k v_t^k\right)\Big) \leqslant c_0\sum_{k=1}^{3}\left|v_t^k\right|^2, \tag{2.8}$$

where

$$c_0 := 1 + \frac{1}{2a^2}\max_{\varepsilon \leqslant |\xi| \leqslant 1/\varepsilon}|\xi|^{4\theta - 2}.$$

Next, we define the desired Lyapunov function $F_{\mathrm{mid}}(v) = F_{\mathrm{mid}}(v)(t,\xi)$ such that

$$F_{\mathrm{mid}}(v) := \frac{1}{c_1} E_{\mathrm{mid}}(v) + \sum_{k=1}^{3} \mathrm{Re}\big(\bar{v}^k v_t^k\big)$$

with a sufficiently small positive constant c_1 to be chosen later.
Taking into consideration (2.7) and (2.8) we get for the evolution partial derivative of $F_{\mathrm{mid}}(v)$ with respect to t the relation

$$\frac{\partial}{\partial t} F_{\mathrm{mid}}(v) \leqslant -\frac{1}{2}\Big(\frac{2\varepsilon^{2\theta}}{c_1} - 2c_0\Big) \sum_{k=1}^{3} \big|v_t^k\big|^2 - \frac{1}{2}|\xi|^2 \sum_{k=1}^{3} \varpi_k \big|v^k\big|^2.$$

There exist positive constants

$$c_2 := \frac{1}{a^2\varepsilon^2} \quad \text{and} \quad c_3 := c_2 + \frac{1}{c_1}$$

satisfying

$$c_2 E_{\mathrm{mid}}(v) \leqslant F_{\mathrm{mid}}(v) \leqslant c_3 E_{\mathrm{mid}}(v)$$

for a sufficiently small constant c_1 to be determined later. We observe that

$$\Big| \sum_{k=1}^{3} \mathrm{Re}\big(\bar{v}^k v_t^k\big) \Big| \leqslant c_2 E_{\mathrm{mid}}(v).$$

Choosing

$$c_1 := \min\Big\{ \frac{2\varepsilon^{2\theta}}{2c_0+1}; \frac{1}{2c_2} \Big\},$$

we get

$$\frac{\partial}{\partial t} F_{\mathrm{mid}}(v) \leqslant -\frac{1}{c_3} F_{\mathrm{mid}}(v). \tag{2.9}$$

The use of Grönwall's inequality in (2.9) and the relationship between $F_{\mathrm{mid}}(v)$ and $E_{\mathrm{mid}}(v)$ lead to an exponential decay estimate for the energy $E_{\mathrm{mid}}(v)$. That shows an exponential stability of solutions for frequencies in the bounded zone. □

2.2.2. Diagonalization procedure for Case 2.1

Theorem 2.2.2. *Let us consider $\theta \in [0,1/2)$ with $\xi \in \mathcal{Z}_{\mathrm{int}}(\varepsilon)$ or $\theta \in (1/2,1]$ with $\xi \in \mathcal{Z}_{\mathrm{ext}}(\varepsilon)$. After ℓ steps of diagonalization procedure the starting system* (2.6) *is transformed to*

$$\begin{cases} D_t W^{(\ell)} - \Big(i|\xi|^{2\theta}\Lambda_1 + \sum\limits_{j=2}^{\ell} \Lambda_j + R_\ell\Big) W^{(\ell)} = 0, & \xi \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ W^{(\ell)}(0,\xi) = W_0^{(\ell)}(\xi), & \xi \in \mathbb{R}^3, \end{cases}$$

with the diagonal matrices $\Lambda_1, \ldots, \Lambda_\ell$ and the remainder R_ℓ. The asymptotic behavior of these matrices can be described as follows:

$$\Lambda_1 = O(1), \quad \Lambda_j = O\big(|\xi|^{2(j-1)+2\theta(3-2j)}\big), \quad R_\ell = O\big(|\xi|^{2\ell-1+4\theta(1-\ell)}\big).$$

Moreover, the characteristic roots $\mu_{\ell,j}=\mu_{\ell,j}(|\xi|)$ with $j=1,\dots,6$ have the following asymptotic behavior:

$$
\begin{aligned}
\mu_{\ell,1}&= ia^2|\xi|^{2-2\theta}+z_1(a)+z_3, & \mu_{\ell,4}&= i|\xi|^{2\theta}-ib^2|\xi|^{2-2\theta}-z_1(a)-z_3,\\
\mu_{\ell,2}&= ia^2|\xi|^{2-2\theta}+z_1(a)+z_5, & \mu_{\ell,5}&= i|\xi|^{2\theta}-ia^2|\xi|^{2-2\theta}-z_1(b)-z_9,\\
\mu_{\ell,3}&= ib^2|\xi|^{2-2\theta}+z_1(b)+z_7, & \mu_{\ell,6}&= i|\xi|^{2\theta}-ia^2|\xi|^{2-2\theta}-z_1(a)-z_5,
\end{aligned}
$$

modulo $O\big(|\xi|^{7-12\theta}\big)$, where $z_1(a),z_1(b)=O\big(|\xi|^{4-6\theta}\big)$, $z_3,z_5,z_7,z_9=O\big(|\xi|^{6-10\theta}\big)$ whose expressions will be explicitly given in the proof.

Proof. First of all, the matrix $\frac{i}{2}|\xi|^{2\theta}B_0$ has a dominant influence to begin the diagonalization procedure in comparison with the matrix $|\xi|B_1$. For this reason, we diagonalize B_0 in the first step. After applying the substitution

$$
W^{(1)}(t,\xi):=T_1^{-1}W^{(0)}(t,\xi):=\sqrt{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 1 & 0\\ 0 & -1 & 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}^{-1} W^{(0)}(t,\xi), \tag{2.10}
$$

we obtain the system

$$
D_tW^{(1)}-i|\xi|^{2\theta}\Lambda_1W^{(1)}-R_1W^{(1)}=0,
$$

where the diagonal matrix Λ_1 is given by

$$
\Lambda_1=\text{diag}(0,0,0,1,1,1)
$$

and the remainder R_1 is given by

$$
R_1=|\xi|T_1^{-1}B_1T_1=|\xi|\begin{pmatrix} 0 & 0 & 0 & a & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & a\\ 0 & 0 & 0 & 0 & b & 0\\ a & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & b & 0 & 0 & 0\\ 0 & a & 0 & 0 & 0 & 0 \end{pmatrix}=O(|\xi|).
$$

We introduce the new matrix $T_2:=I_{6\times 6}+\mathcal{N}_2$ in which $\mathcal{N}_2=\mathcal{N}_2(|\xi|)$ will be defined later. Similarly, we set

$$
W^{(2)}(t,\xi):=T_2^{-1}W^{(1)}(t,\xi)
$$

to get

$$
D_tW^{(2)}-i|\xi|^{2\theta}T_2^{-1}\Lambda_1T_2W^{(2)}-T_2^{-1}R_1T_2W^{(2)}=0. \tag{2.11}
$$

Remarking the symbol $[\Lambda_1,T_2]:=\Lambda_1T_2-T_2\Lambda_1$ we get

$$
D_tW^{(2)}-i|\xi|^{2\theta}\Lambda_1W^{(2)}-i|\xi|^{2\theta}T_2^{-1}[\Lambda_1,T_2]W^{(2)}-T_2^{-1}R_1W^{(2)}-T_2^{-1}R_1\mathcal{N}_2W^{(2)}=0, \tag{2.12}
$$

and it implies

$$
\begin{aligned}
D_tW^{(2)}-i|\xi|^{2\theta}\Lambda_1W^{(2)}-T_2^{-1}\big(i|\xi|^{2\theta}[\Lambda_1,\mathcal{N}_2]+R_1\big)W^{(2)}&\\
-R_1\mathcal{N}_2W^{(2)}+\mathcal{N}_2T_2^{-1}R_1\mathcal{N}_2W^{(2)}&=0.
\end{aligned} \tag{2.13}
$$

In the above we used $[\Lambda_1, T_2] = [\Lambda_1, \mathcal{N}_2]$ and $T_2^{-1} = I_{6\times 6} - \mathcal{N}_2 T_2^{-1}$. We define the matrix

$$\mathcal{N}_2(|\xi|) := i|\xi|^{1-2\theta} \begin{pmatrix} 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 & -b & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \end{pmatrix} = O\big(|\xi|^{1-2\theta}\big). \tag{2.14}$$

Then, the next relation holds

$$i|\xi|^{2\theta}[\Lambda_1, \mathcal{N}_2] + R_1 = 0.$$

Also, we know

$$\Lambda_2 = R_1 \mathcal{N}_2 = |\xi|^{2-2\theta} \mathrm{diag}\big(ia^2, ia^2, ib^2, -ib^2, -ia^2, -ia^2\big) = O\big(|\xi|^{2-2\theta}\big).$$

Thus, we derive

$$D_t W^{(2)} - i|\xi|^{2\theta}\Lambda_1 W^{(2)} - \Lambda_2 W^{(2)} - R_2 W^{(2)} = 0. \tag{2.15}$$

The remainder R_2 is represented by

$$R_2 = -\mathcal{N}_2 T_2^{-1} R_1 \mathcal{N}_2 = \begin{pmatrix} z_1(a) & 0 & 0 & z_2(a) & 0 & 0 \\ 0 & z_1(a) & 0 & 0 & 0 & z_2(a) \\ 0 & 0 & z_1(b) & 0 & z_2(b) & 0 \\ z_2(a) & 0 & 0 & -z_1(a) & 0 & 0 \\ 0 & 0 & z_2(b) & 0 & -z_1(b) & 0 \\ 0 & z_2(a) & 0 & 0 & 0 & -z_1(a) \end{pmatrix} = O\big(|\xi|^{3-4\theta}\big),$$

with $z_1(y)$ and $z_2(y)$ for $y = a, b$ such that

$$z_1(y) = \frac{iy^4|\xi|^{4-6\theta}}{1 - y^2|\xi|^{2-4\theta}} = O\big(|\xi|^{4-6\theta}\big) \quad \text{and} \quad z_2(y) = \frac{y^3|\xi|^{3-4\theta}}{1 - y^2|\xi|^{2-4\theta}} = O\big(|\xi|^{3-4\theta}\big).$$

Similarly, we introduce $T_3 := I_{6\times 6} + \mathcal{N}_3$ and

$$W^{(3)}(t,\xi) := T_3^{-1} W^{(2)}(t,\xi)$$

in which $\mathcal{N}_3 = \mathcal{N}_3(|\xi|)$ is defined by

$$\mathcal{N}_3(|\xi|) := -\frac{i}{|\xi|^{2\theta}} \begin{pmatrix} 0 & 0 & 0 & z_2(a) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_2(a) \\ 0 & 0 & 0 & 0 & z_2(b) & 0 \\ -z_2(a) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -z_2(b) & 0 & 0 & 0 \\ 0 & -z_2(a) & 0 & 0 & 0 & 0 \end{pmatrix} = O\big(|\xi|^{3-6\theta}\big). \tag{2.16}$$

By applying the previous method it yields

$$D_t W^{(3)} - i|\xi|^{2\theta}\Lambda_1 W^{(3)} - \Lambda_2 W^{(3)} - \Lambda_3 W^{(3)} - R_3 W^{(3)} = 0,$$

where the diagonal matrix Λ_3 is given by

$$\Lambda_3 = \mathrm{diag}(z_1(a), z_1(a), z_1(b), -z_1(a), -z_1(b), -z_1(a)) = O\big(|\xi|^{4-6\theta}\big)$$

and the remainder R_3 is given by

$$R_3 = T_3^{-1}[\Lambda_2, \mathcal{N}_3] + T_3^{-1} R_2 \mathcal{N}_3 - \mathcal{N}_3 T_3^{-1} \Lambda_3 = O\big(|\xi|^{5-8\theta}\big).$$

From direct calculations, the remainder can be represented by

$$R_3 = \begin{pmatrix} z_3 & 0 & 0 & z_4 & 0 & 0 \\ 0 & z_5 & 0 & 0 & 0 & z_6 \\ 0 & 0 & z_7 & 0 & z_8 & 0 \\ z_4 & 0 & 0 & -z_3 & 0 & 0 \\ 0 & 0 & z_{10} & 0 & -z_9 & 0 \\ 0 & z_6 & 0 & 0 & 0 & -z_5 \end{pmatrix} = O\big(|\xi|^{5-8\theta}\big),$$

where the elements of the above matrix are

$$\begin{aligned}
z_3 &= \frac{1}{|\xi|^{4\theta} - z_2^2(a)}\big(i\big(a^2+b^2\big)z_2^2(a)|\xi|^{2-2\theta} + 2z_1(a)z_2^2(a) + i|\xi|^{2\theta}z_2^2(a)\big),\\
z_4 &= \frac{1}{|\xi|^{4\theta} - z_2^2(a)}\big(\big(a^2+b^2\big)|\xi|^2 z_2(a) - 2i|\xi|^{2\theta}z_1(a)z_2(a) + z_2^3(a)\big),\\
z_5 &= \frac{1}{|\xi|^{4\theta} - z_2^2(a)}\big(2ia^2|\xi|^{2-2\theta}z_2^2(a) + i|\xi|^{2\theta}z_2^2(a) + 2z_1(a)z_2^2(a)\big),\\
z_6 &= \frac{1}{|\xi|^{4\theta} - z_2^2(a)}\big(2a^2|\xi|^2 z_2(a) - 2i|\xi|^{2\theta}z_1(a)z_2(a) + z_2^3(a)\big),\\
z_7 &= \frac{1}{|\xi|^{4\theta} - z_2^2(b)}\big(i|\xi|^{2\theta}z_2^2(b) + z_2^3(b) + z_1(b)z_2^2(b) - i\big(a^2+b^2\big)|\xi|^{2-2\theta}z_2^2(b)\big),\\
z_8 &= \frac{1}{|\xi|^{4\theta} - z_2^2(b)}\big(\big(a^2+b^2\big)|\xi|^2 z_2(b) + z_2^2(b) - 2i|\xi|^{2\theta}z_1(b)z_2(b)\big),\\
z_9 &= \frac{1}{|\xi|^{4\theta} - z_2^2(b)}\big(2z_1(b)z_2^2(b) + i|\xi|^{2\theta}z_2^2(b) + i\big(a^2+b^2\big)|\xi|^{2-2\theta}z_2^2(b)\big),\\
z_{10} &= \frac{1}{|\xi|^{4\theta} - z_2^2(b)}\big(\big(a^2+b^2\big)|\xi|^2 z_2(b) + z_2^3(b) - i|\xi|^{2\theta}z_2^2(b) - i|\xi|^{2\theta}z_1(b)z_2(b)\big).
\end{aligned}$$

The asymptotic behavior of the elements are as follows:

$$z_3, z_5, z_7, z_9 = O\big(|\xi|^{6-10\theta}\big) \quad \text{and} \quad z_4, z_6, z_8, z_{10} = O\big(|\xi|^{5-8\theta}\big).$$

Indeed, setting $T_4 := I_{6\times 6} + \mathcal{N}_4$ and

$$W^{(4)}(t,\xi) := T_4^{-1}W^{(3)}(t,\xi)$$

where $\mathcal{N}_4 = \mathcal{N}_4(|\xi|)$ is defined by

$$\mathcal{N}_4(|\xi|) := -\frac{i}{|\xi|^{2\theta}} \begin{pmatrix} 0 & 0 & 0 & z_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_6 \\ 0 & 0 & 0 & 0 & z_8 & 0 \\ -z_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -z_{10} & 0 & 0 & 0 \\ 0 & -z_6 & 0 & 0 & 0 & 0 \end{pmatrix} = O\big(|\xi|^{5-10\theta}\big), \tag{2.17}$$

we get the system

$$D_t W^{(4)} - i|\xi|^{2\theta}\Lambda_1 W^{(4)} - \Lambda_2 W^{(4)} - \Lambda_3 W^{(4)} - \Lambda_4 W^{(4)} - R_4 W^{(4)} = 0.$$

Here the diagonal matrix Λ_4 is given by

$$\Lambda_4 = \mathrm{diag}(z_3, z_5, z_7, -z_3, -z_9, -z_5) = O\big(|\xi|^{6-10\theta}\big)$$

and the remainder R_4 is given by

$$R_4 = T_4^{-1}[\Lambda_2 + \Lambda_3, \mathcal{N}_4] - \mathcal{N}_4 T_4^{-1}\Lambda_4 = O\big(|\xi|^{7-12\theta}\big).$$

In conclusion, after four steps of diagonalization, we transfer the system in the Cauchy problem (2.6) to

$$D_tW^{(4)} - (i|\xi|^{2\theta}\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4)W^{(4)} - R_4W^{(4)} = 0,$$

with diagonal matrices $\Lambda_1, \dots, \Lambda_4$ and with remainder R_4. The asymptotic behavior of above matrices can be described as follows:

$$\Lambda_1 = O(1), \quad \Lambda_2 = O\big(|\xi|^{2-2\theta}\big), \quad \Lambda_3 = O\big(|\xi|^{4-6\theta}\big), \quad \Lambda_4 = O\big(|\xi|^{6-10\theta}\big), \quad R_4 = O\big(|\xi|^{7-12\theta}\big).$$

Lastly, one may carry out further steps of diagonalization proposed in the papers [89], [110] to complete the proof. □

To end this subsection, we denote the matrices $T_{\theta,\text{int}} = T_{\theta,\text{int}}(|\xi|)$ and $T_{\theta,\text{ext}} = T_{\theta,\text{ext}}(|\xi|)$ by:

$$\begin{aligned} T_{\theta,\text{int}} &:= T_1(I_{6\times 6} + \mathcal{N}_2(|\xi|))(I_{6\times 6} + \mathcal{N}_3(|\xi|))(I_{6\times 6} + \mathcal{N}_4(|\xi|)) \quad \text{if } \theta \in \big[0, \tfrac{1}{2}\big), \\ T_{\theta,\text{ext}} &:= T_1(I_{6\times 6} + \mathcal{N}_2(|\xi|))(I_{6\times 6} + \mathcal{N}_3(|\xi|))(I_{6\times 6} + \mathcal{N}_4(|\xi|)) \quad \text{if } \theta \in \big(\tfrac{1}{2}, 1\big], \end{aligned}$$

where $T_1, \mathcal{N}_2(|\xi|), \mathcal{N}_3(|\xi|)$ and $\mathcal{N}_4(|\xi|)$ are defined in (2.10), (2.14), (2.16) and (2.17), respectively.

2.2.3. Diagonalization procedure for Case 2.2

Theorem 2.2.3. *Let us consider $\theta \in [0, 1/2)$ with $\xi \in \mathcal{Z}_{\text{ext}}(\varepsilon)$ or $\theta \in (1/2, 1]$ with $\xi \in \mathcal{Z}_{\text{int}}(\varepsilon)$. After ℓ steps of diagonalization procedure the starting system* (2.6) *is transformed to*

$$\begin{cases} D_tW^{(\ell)} - \Big(\Lambda_1 + \Lambda_2 + \sum\limits_{j=3}^{\ell} \Lambda_j + R_\ell\Big)W^{(\ell)} = 0, & \xi \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ W^{(\ell)}(0,\xi) = W_0^{(\ell)}(\xi), & \xi \in \mathbb{R}^3, \end{cases}$$

with the diagonal matrices $\Lambda_1, \dots, \Lambda_\ell$ and the remainder R_ℓ. The asymptotic behavior of above matrices can be described as follows:

$$\Lambda_1 = O(|\xi|), \quad \Lambda_2 = O\big(|\xi|^{2\theta}\big), \quad \Lambda_j = O\big(|\xi|^{(2\theta-1)(2j-5)+2\theta}\big), \quad R_\ell = O\big(|\xi|^{2(2\theta-1)(\ell-2)+2\theta}\big).$$

Moreover, the characteristic roots $\mu_{\ell,j} = \mu_{\ell,j}(|\xi|)$ with $j = 1, \dots, 6$ have the following asymptotic behavior:

$$\mu_{\ell,j} = \pm|\xi|y - \frac{i}{2}|\xi|^{2\theta} - \frac{1}{8y}|\xi|^{4\theta-1}$$

modulo $O\big(|\xi|^{6\theta-2}\big)$, where $y = a$ if $j = 1, 2, 4, 5$ and $y = b$ if $j = 3, 6$. Additionally, when $j = 4, 5, 6$, we take in the first term the negative sign and when $j = 1, 2, 3$ we take in the first term the positive sign.

Proof. To begin with, the diagonal matrix $|\xi|B_1$ has a dominant influence to begin the diagonalization procedure in comparison with the matrix $\frac{i}{2}|\xi|^{2\theta}B_0$. For this reason, we define $T_1 := I_{6\times 6}$ and we shall diagonalize B_1 modulo a new remainder. Applying the substitution

$$W^{(2)}(t,\xi) := T_2^{-1}W^{(0)}(t,\xi),$$

where $T_2 := I_{6\times 6} + \mathcal{N}_2$ and $\mathcal{N}_2 = \mathcal{N}_2(|\xi|)$ will be defined later. Analogously to the proof of (2.11) to (2.13), the following system comes into play:

$$D_tW^{(2)} - T_2^{-1}\Lambda_1T_2W^{(2)} - \frac{i}{2}|\xi|^{2\theta}T_2^{-1}B_0T_2W^{(2)} = 0, \tag{2.18}$$

where the diagonal matrix Λ_1 is given by

$$\Lambda_1 = |\xi|B_1 = |\xi|\text{diag}(a, a, b, -a, -a, -b) = O(|\xi|).$$

Similar as the proof in Section 2.2.2, we rewrite (2.18) as follows:

$$D_tW^{(2)} - \Lambda_1 W^{(2)} - T_2^{-1}\Big([\Lambda_1, \mathcal{N}_2] + \frac{i}{2}|\xi|^{2\theta}B_0\Big)W^{(2)} - \frac{i}{2}|\xi|^{2\theta}T_2^{-1}B_0\mathcal{N}_2W^{(2)} = 0.$$

Introducing the matrix

$$\mathcal{N}_2(|\xi|) := \frac{i|\xi|^{2\theta-1}}{4}\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{a} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{b} \\ \frac{1}{a} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{b} & 0 & 0 & 0 \end{pmatrix} = O\big(|\xi|^{2\theta-1}\big), \tag{2.19}$$

we derive

$$\Lambda_2 = [\Lambda_1, \mathcal{N}_2] + \frac{i}{2}|\xi|^{2\theta}B_0 = \frac{i}{2}|\xi|^{2\theta}\mathrm{diag}(1,1,1,1,1,1) = O\big(|\xi|^{2\theta}\big).$$

In fact, according to $T_2^{-1} = I_{6\times 6} - \mathcal{N}_2T_2^{-1}$ we have

$$D_tW^{(2)} - \Lambda_1W^{(2)} - \Lambda_2W^{(2)} - R_2W^{(2)} = 0,$$

where the remainder R_2 is given by

$$R_2 = -\mathcal{N}_2T_2^{-1}\Lambda_2 + \frac{i}{2}|\xi|^{2\theta}T_2^{-1}B_0\mathcal{N}_2 = O\big(|\xi|^{4\theta-1}\big).$$

Thus, the decomposition of the remainder R_2 leads to

$$R_2 = -\Big(\mathcal{N}_2\Lambda_2 - \frac{i}{2}|\xi|^{2\theta}B_0\mathcal{N}_2\Big) + \mathcal{N}_2\mathcal{N}_2T_2^{-1}\Lambda_2 - \frac{i}{2}|\xi|^{2\theta}\mathcal{N}_2T_2^{-1}B_0\mathcal{N}_2 = \Lambda_3 + R_3,$$

where the diagonal matrix is given by

$$\Lambda_3 = -\frac{1}{8}|\xi|^{4\theta-1}\mathrm{diag}\Big(\frac{1}{a}, \frac{1}{a}, \frac{1}{b}, -\frac{1}{a}, -\frac{1}{a}, -\frac{1}{b}\Big) = O\big(|\xi|^{4\theta-1}\big).$$

Furthermore, we decompose the remainder R_3 when $\ell \geqslant 3$ as follows:

$$\begin{aligned} R_3 &= \mathcal{N}_2^2T_2^{-1}\Lambda_2 - \frac{i}{2}|\xi|^{2\theta}\mathcal{N}_2T_2^{-1}B_0\mathcal{N}_2 \\ &= -\mathcal{N}_2\Lambda_3 + \Big(\frac{i}{2}|\xi|^{2\theta}\mathcal{N}_2^2T_2^{-1}B_0\mathcal{N}_2 - \mathcal{N}_2^3T_2^{-1}\Lambda_2\Big) \\ &= -\mathcal{N}_2\Lambda_3 + \mathcal{N}_2^2\Lambda_3 - \mathcal{N}_2^3\Lambda_3 + \cdots + (-\mathcal{N}_2)^{2(\ell-3)} + R_{2\ell-3} \\ &= G_4 + \Lambda_4 + \cdots + G_{\ell-3} + \Lambda_{\ell-3} + R_{2\ell-3}. \end{aligned}$$

Here the remainder is

$$R_{2\ell-3} = (-1)^{2\ell-5}\Big(\frac{i}{2}|\xi|^{2\theta}\mathcal{N}_2^{2\ell-5}T_2^{-1}B_0\mathcal{N}_2 - \mathcal{N}_2^{2(\ell-2)}T_2^{-1}\Lambda_2\Big) = O\big(|\xi|^{2(2\theta-1)(\ell-2)+2\theta}\big).$$

Thus, the expansion of R_3 leads to

$$D_tW^{(2)} - \Lambda_1W^{(2)} - \Lambda_2W^{(2)} - \Lambda_3W^{(2)} - (G_4 + \Lambda_4 + \cdots + G_{\ell-3} + \Lambda_{\ell-3} + R_{2\ell-3})W^{(2)} = 0.$$

Here $\Lambda_4, \Lambda_5, \ldots, \Lambda_{[\ell/2]}$ are the diagonal matrices and $G_4, G_5, \ldots, G_{[\ell/2]}$ have the same structure. To be more concrete, the diagonal matrices Λ_j for $j = 4, 5, \ldots, \ell$ can be shown by

$$\Lambda_j = \mathcal{N}_2^{2(j-3)}\Lambda_3 = \mathrm{diag}(z_{11}(a,j), z_{11}(a,j), z_{11}(b,j), -z_{11}(a,j), -z_{11}(a,j), -z_{11}(b,j)).$$

We observe that the values of elements of first, second positions in Λ_j are the same and the value of elements of forth, fifth positions also are the same. The matrices $G_j = (-\mathcal{N}_2)^{2(j-3)-1}\Lambda_3$ satisfying

$$G_j = \begin{pmatrix} 0 & 0 & 0 & z_{12}(a,j) & 0 & 0 \\ 0 & 0 & 0 & 0 & z_{12}(a,j) & 0 \\ 0 & 0 & 0 & 0 & 0 & z_{12}(b,j) \\ z_{12}(a,j) & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{12}(a,j) & 0 & 0 & 0 & 0 \\ 0 & 0 & z_{12}(b,j) & 0 & 0 & 0 \end{pmatrix},$$

where the elements are represented by

$$\begin{aligned} z_{11}(y,j) &= \frac{|\xi|^{2\theta}}{2}\Big(\frac{|\xi|^{2\theta-1}}{4y}\Big)^{2(j-3)+1} = O\big(|\xi|^{(2\theta-1)(2j-5)+2\theta}\big), \\ z_{12}(y,j) &= \frac{i|\xi|^{2\theta}}{2}\Big(\frac{|\xi|^{2\theta-1}}{16y^2}\Big)^{2(j-3)} = O\big(|\xi|^{2(2\theta-1)(j-3)+2\theta}\big), \end{aligned}$$

for $y = a, b$. We may use the same method as previous subsection to diagonalize the matrices G_j. One may carry out further steps of diagonalization proposed in the papers [89], [110] to complete the proof. $\square$

To end this subsection, we denote the matrices $T_{\theta,\text{int}} = T_{\theta,\text{int}}(|\xi|)$ and $T_{\theta,\text{ext}} = T_{\theta,\text{ext}}(|\xi|)$ by

$$\begin{aligned} T_{\theta,\text{int}} &:= I_{6\times6} + \mathcal{N}_2(|\xi|) \quad \text{if } \theta \in \big(\tfrac{1}{2}, 1\big], \\ T_{\theta,\text{ext}} &:= I_{6\times6} + \mathcal{N}_2(|\xi|) \quad \text{if } \theta \in \big[0, \tfrac{1}{2}\big), \end{aligned}$$

where $\mathcal{N}_2(|\xi|)$ is defined in (2.19).

2.2.4. Diagonalization procedure for Case 2.3

In this case when $\theta = 1/2$, the matrices $\frac{i}{2}|\xi|^{2\theta}B_0$ and $|\xi|B_1$ have the same influence on the principal part. For this reason, we apply directly the diagonalization procedure to the system

$$D_t W^{(0)} - |\xi|\Big(\frac{i}{2}B_0 + B_1\Big)W^{(0)} = 0 \tag{2.20}$$

as a whole. To study the eigenvalues we recall the relation

$$\det\Big(\Big(\frac{i}{2}B_0 + B_1\Big) - \lambda I_{6\times6}\Big) = \big(\lambda^2 - i\lambda - a^2\big)^2\big(\lambda^2 - i\lambda - b^2\big) = 0.$$

Hence, we study the solutions of the two quadratic equations

$$\lambda^2 - i\lambda - a^2 = 0, \tag{2.21}$$

$$\lambda^2 - i\lambda - b^2 = 0. \tag{2.22}$$

Let us consider the equation (2.21). Setting $\lambda = \text{Re}\lambda + i\text{Im}\lambda$ with real numbers $\text{Re}\lambda$ and $\text{Im}\lambda$ gives the system of equations, respectively,

$$\begin{cases} (\text{Re}\lambda)^2 - (\text{Im}\lambda)^2 + \text{Im}\lambda - a^2 = 0, \\ 2(\text{Re}\lambda)(\text{Im}\lambda) - \text{Re}\lambda = 0. \end{cases}$$

Now, we distinguish the solution between two cases.

Case 2.3.1: If $\text{Re}\lambda = 0$, then $(\text{Im}\lambda)_{1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4a^2}$.
Case 2.3.2: If $\text{Im}\lambda = \frac{1}{2}$, then $(\text{Re}\lambda)_{1,2} = \pm\frac{1}{2}\sqrt{4a^2-1}$.

If $a^2 = 1/4$, then $\mathrm{Re}\lambda = 0$ and $(\mathrm{Im}\lambda)_{1,2} = \frac{1}{2}$; if $a^2 \in (0,1/4)$, then $\mathrm{Re}\lambda = 0$ and $(\mathrm{Im}\lambda)_{1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4a^2}$; if $a^2 \in (1/4,\infty)$, then $(\mathrm{Re}\lambda)_{1,2} = \pm\frac{1}{2}\sqrt{4a^2-1}$ and $\mathrm{Im}\lambda = \frac{1}{2}$. One can follow the above discussion to study the equation (2.22) with respect to b.

However, it is impossible to verify that all eigenvalues are pairwise distinct for all $\xi \in \mathbb{R}^3\backslash\{0\}$. Hence, the matrix $|\xi|\big(\frac{i}{2}B_0 + B_1\big)$ cannot be completely diagonalized. Consequently, Jordan normal forms come into play. After a change of variables, we obtain "almost diagonalized" systems.

Theorem 2.2.4. *Let us consider $\theta = 1/2$. If $b^2 > a^2 > 0$, $b^2 \neq 1/4$, $a^2 \neq 1/4$, then there exist eigenvalues $\lambda_{1,2} = \big(\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4a^2}\big)i$ when $a^2 \in (0,1/4)$, $\lambda_{1,2} = \frac{1}{2}\sqrt{4a^2-1} + \frac{i}{2}$ when $a^2 \in (1/4,\infty)$, $\lambda_{3,4} = \big(\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4b^2}\big)i$ when $b^2 \in (0,1/4)$ and $\lambda_{3,4} = \pm\frac{1}{2}\sqrt{4b^2-1} + \frac{i}{2}$ when $b^2 \in (1/4,\infty)$. The system* (2.20) *can be transformed to*

$$D_t W^{(1)} - |\xi|(J_2(\lambda_1) \oplus J_2(\lambda_2) \oplus J_1(\lambda_3) \oplus J_1(\lambda_4))W^{(1)} = 0. \tag{2.23}$$

A representation of solutions to the system (2.23) *is*

$$\begin{aligned}
W_1^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_1 t}\big(W_{0,1}^{(1)}(\xi) + i|\xi|tW_{0,2}^{(1)}(\xi)\big), &\quad W_2^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_1 t}W_{0,2}^{(1)}(\xi),\\
W_3^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_2 t}\big(W_{0,3}^{(1)}(\xi) + i|\xi|tW_{0,4}^{(1)}(\xi)\big), &\quad W_4^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_2 t}W_{0,4}^{(1)}(\xi),\\
W_5^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_3 t}W_{0,5}^{(1)}(\xi), &\quad W_6^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_4 t}W_{0,6}^{(1)}(\xi).
\end{aligned}$$

Furthermore, if $1/4 = b^2 > a^2 > 0$, then there exist eigenvalues $\lambda_{1,2} = i\big(\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4a^2}\big)$ and $\lambda_3 = \frac{i}{2}$, such that the system (2.20) *can be transformed to*

$$D_t W^{(1)} - |\xi|(J_2(\lambda_1) \oplus J_2(\lambda_2) \oplus J_2(\lambda_3))W^{(1)} = 0. \tag{2.24}$$

A representation of solutions to the system (2.24) *is*

$$\begin{aligned}
W_1^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_1 t}\big(W_{0,1}^{(1)}(\xi) + i|\xi|tW_{0,2}^{(1)}(\xi)\big), &\quad W_2^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_1 t}W_{0,2}^{(1)}(\xi),\\
W_3^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_2 t}\big(W_{0,3}^{(1)}(\xi) + i|\xi|tW_{0,4}^{(1)}(\xi)\big), &\quad W_4^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_2 t}W_{0,4}^{(1)}(\xi),\\
W_5^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_3 t}\big(W_{0,5}^{(1)}(\xi) + i|\xi|tW_{0,6}^{(1)}(\xi)\big), &\quad W_6^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_3 t}W_{0,6}^{(1)}(\xi).
\end{aligned}$$

In the case $b^2 > a^2 = 1/4$, there exist eigenvalues $\lambda_1 = \frac{i}{2}$ and $\lambda_{2,3} = \pm\frac{1}{2}\sqrt{4b^2-1} + \frac{i}{2}$ and the system (2.20) *can be transformed to*

$$D_t W^{(1)} - |\xi|(J_4(\lambda_1) \oplus J_1(\lambda_2) \oplus J_1(\lambda_3))W^{(1)} = 0. \tag{2.25}$$

A representation of solutions to the system (2.25) *is*

$$\begin{aligned}
W_1^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_1 t}\Big(W_{0,1}^{(1)}(\xi) + i|\xi|tW_{0,2}^{(1)}(\xi) - \frac{1}{2}|\xi|^2t^2W_{0,3}^{(1)}(\xi) - \frac{i}{6}|\xi|^3t^3W_{0,4}^{(1)}(\xi)\Big),\\
W_2^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_1 t}\Big(W_{0,2}^{(1)}(\xi) + i|\xi|tW_{0,3}^{(1)}(\xi) - \frac{1}{2}|\xi|^2t^2W_{0,4}^{(1)}(\xi)\Big),\\
W_3^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_1 t}\Big(W_{0,3}^{(1)}(\xi) + i|\xi|tW_{0,4}^{(1)}(\xi)\Big), \qquad W_4^{(1)}(t,\xi) = \mathrm{e}^{i|\xi|\lambda_1 t}W_{0,4}^{(1)}(\xi),\\
W_5^{(1)}(t,\xi) &= \mathrm{e}^{i|\xi|\lambda_2 t}W_{0,5}^{(1)}(\xi), \qquad W_6^{(1)}(t,\xi) = \mathrm{e}^{i|\xi|\lambda_3 t}W_{0,6}^{(1)}(\xi).
\end{aligned}$$

Proof. The proofs of these statements are based on the application of the diagonalization procedure and a careful treatment of the Jordan matrices appearing in the systems (see the proof of Theorem 7.6 in [88]). $\square$

2.2.5. Representations of solutions

According to Theorems 2.2.2 and 2.2.3, we may obtain results for representations of solutions. Before doing this, we introduce the following notation:

$$\operatorname{diag}\big(\mathrm{e}^{-\mu_l(|\xi|)t}\big)_{l=1}^{6} := \operatorname{diag}\big(\mathrm{e}^{-\mu_1(|\xi|)t}, \mathrm{e}^{-\mu_2(|\xi|)t}, \mathrm{e}^{-\mu_3(|\xi|)t}, \mathrm{e}^{-\mu_4(|\xi|)t}, \mathrm{e}^{-\mu_5(|\xi|)t}, \mathrm{e}^{-\mu_6(|\xi|)t}\big).$$

We now state our main results.

Theorem 2.2.5. *There exists a matrix $T_{\theta,\mathrm{int}} = T_{\theta,\mathrm{int}}(|\xi|)$ for $\theta \in [0,1/2)\cup(1/2,1]$, which is uniformly invertible for small frequencies such that the following representation formula holds:*

$$W^{(0)}(t,\xi) = T^{-1}_{\theta,\mathrm{int}}(|\xi|)\operatorname{diag}\left(\mathrm{e}^{-\mu_l(|\xi|)t}\right)_{l=1}^{6} T_{\theta,\mathrm{int}}(|\xi|)W_0^{(0)}(\xi),$$

where the characteristic roots $\mu_l = \mu_l(|\xi|)$ have the following asymptotic behavior:

- *when $\theta \in [0,1/2)$:*

$$\begin{aligned}
\mu_1(|\xi|)&= a^2|\xi|^{2-2\theta} - iz_1(a) - iz_2, & \mu_4(|\xi|)&= |\xi|^{2\theta} - b^2|\xi|^{2-2\theta} + iz_1(a) + iz_2,\\
\mu_2(|\xi|)&= a^2|\xi|^{2-2\theta} - iz_1(a) - iz_3, & \mu_5(|\xi|)&= |\xi|^{2\theta} - a^2|\xi|^{2-2\theta} + iz_1(b) + iz_5,\\
\mu_3(|\xi|)&= b^2|\xi|^{2-2\theta} - iz_1(b) - iz_4, & \mu_6(|\xi|)&= |\xi|^{2\theta} - a^2|\xi|^{2-2\theta} + iz_1(a) + iz_3,
\end{aligned}$$

  modulo $O\big(|\xi|^{7-12\theta}\big)$;

- *when $\theta \in (1/2,1]$: $y = a$ as $l = 1,2,4,5$ and $y = b$ as $l = 3,6$; when $l = 1,2,3$, we take in the first term the negative sign and when $l = 4,5,6$, we take in the first term the positive sign in*

$$\mu_l(|\xi|) = \pm i|\xi|y + \frac{1}{2}|\xi|^{2\theta} + \frac{i}{8y}|\xi|^{4\theta-1}$$

  modulo $O\big(|\xi|^{6\theta-2}\big)$.

Proof. The statements of Theorems 2.2.2 and 2.2.3, as well as the structure of the matrices $T_{\theta,\mathrm{int}}(|\xi|)$ and $T^{-1}_{\theta,\mathrm{int}}(|\xi|)$ allow to get the above representation of solutions. □

Theorem 2.2.6. *There exists a matrix $T_{\theta,\mathrm{ext}} = T_{\theta,\mathrm{ext}}(|\xi|)$ for $\theta \in [0,1/2)\cup(1/2,1]$, which is uniformly invertible for large frequencies such that the following representation formula holds:*

$$W^{(0)}(t,\xi) = T^{-1}_{\theta,\mathrm{ext}}(|\xi|)\operatorname{diag}\left(\mathrm{e}^{-\mu_l(|\xi|)t}\right)_{l=1}^{6} T_{\theta,\mathrm{ext}}(|\xi|)W_0^{(0)}(\xi)$$

and the characteristic roots $\mu_l = \mu_l(|\xi|)$ have the following asymptotic behavior:

- *when $\theta \in [0,1/2)$: $y = a$ as $l = 1,2,4,5$ and $y = b$ as $l = 3,6$; when $l = 1,2,3$, we take in the first term the negative sign and when $l = 4,5,6$, we take in the first term the positive sign in*

$$\mu_l(|\xi|) = \pm i|\xi|y + \frac{1}{2}|\xi|^{2\theta} + \frac{i}{8y}|\xi|^{4\theta-1}$$

  modulo $O\big(|\xi|^{6\theta-2}\big)$;

- *when $\theta \in (1/2,1]$:*

$$\begin{aligned}
\mu_1(|\xi|)&= a^2|\xi|^{2-2\theta} - iz_1(a) - iz_2, & \mu_4(|\xi|)&= |\xi|^{2\theta} - b^2|\xi|^{2-2\theta} + iz_1(a) + iz_2,\\
\mu_2(|\xi|)&= a^2|\xi|^{2-2\theta} - iz_1(a) - iz_3, & \mu_5(|\xi|)&= |\xi|^{2\theta} - a^2|\xi|^{2-2\theta} + iz_1(b) + iz_5,\\
\mu_3(|\xi|)&= b^2|\xi|^{2-2\theta} - iz_1(b) - iz_4, & \mu_6(|\xi|)&= |\xi|^{2\theta} - a^2|\xi|^{2-2\theta} + iz_1(a) + iz_3,
\end{aligned}$$

  modulo $O\big(|\xi|^{7-12\theta}\big)$.

Proof. The statements of Theorems 2.2.2 and 2.2.3, as well as the structure of the matrices $T_{\theta,\mathrm{ext}}(|\xi|)$ and $T^{-1}_{\theta,\mathrm{ext}}(|\xi|)$ allow to get the above representations of solutions. □

2.3. Smoothing effect and propagation of singularities

To formulate the result for smoothing effect of solutions, we introduce the Gevrey spaces $\Gamma^\kappa\big(\mathbb{R}^n\big)$, $\kappa \in [1,\infty)$, where

$$\Gamma^\kappa\big(\mathbb{R}^n\big) := \Big\{ f \in L^2\big(\mathbb{R}^n\big) : \text{ there exists a positive constant } c \text{ such that} \exp\Big(c\langle\xi\rangle^{\frac{1}{\kappa}}\Big)\hat{f} \in L^2\big(\mathbb{R}^n\big)\Big\}.$$

One may find different definitions of Gevrey spaces in [87].

Theorem 2.3.1. *Let us consider the Cauchy problem* (2.1) *with* $\theta \in (0,1)$. *Initial data is supposed to belong to the energy space, that is* $\big(u_0^k, u_1^k\big) \in \dot{H}^1\big(\mathbb{R}^3\big) \times L^2\big(\mathbb{R}^3\big)$ *for* $k = 1,2,3$. *Then, the property of Gevrey smoothing to the solutions appears. This means, that the solutions have the following property:*

$$|D|^{s+1}u^k(t,\cdot), |D|^s u_t^k(t,\cdot) \in \Gamma^\kappa\big(\mathbb{R}^3\big) \quad \textit{for all} \quad s \geqslant 0 \quad \textit{and} \quad t > 0,$$

with the constant $\kappa = \frac{1}{2\min\{1-\theta;\theta\}}$.

Proof. To understand Gevrey smoothing of the solutions, we only need to study the regularity properties of the solutions for frequencies in the zone $\mathcal{Z}_{\text{ext}}(\varepsilon)$. From Theorem 2.2.6, we may estimate

$$\chi_{\text{ext}}(\xi)|\xi|^s|W^{(0)}(t,\xi)| \lesssim \chi_{\text{ext}}(\xi)|\xi|^s \mathrm{e}^{-c|\xi|^{2\min\{1-\theta;\theta\}}t}|W_0^{(0)}(\xi)|.$$

Taking the parameter in Gevrey spaces $\Gamma^\kappa\big(\mathbb{R}^3\big)$ such that $\kappa = \frac{1}{2\min\{1-\theta;\theta\}}$ for $\theta \in (0,1)$, we get

$$\mathcal{F}^{-1}_{\xi\to x}\big(|\xi|^s W^{(0)}(t,\xi)\big)(t,\cdot) \in \big(\Gamma^\kappa\big(\mathbb{R}^3\big)\big)^3 \quad \text{for} \quad t > 0.$$

Then, following the paper [88] one can prove Theorem 2.3.1.
We point out that the case when $\theta = 1/2$ yields, in particular, analytic smoothing. Hence, we expect some better behaviors of solutions in the case of the model (2.1) with structural damping term $(-\Delta)^{1/2}u_t$. It is clear that Gevrey smoothing excludes the property of propagation of $\mathcal{C}^\infty$-singularities. □

If we consider the three dimensional linear elastic waves with friction or viscoelastic damping, then the property of propagation of H^s-singularities of solutions could be of interest. Let us recall the definition of the space $H^s_{\text{loc}}(\{x_0\})$. A function f belongs to $H^s_{\text{loc}}(\{x_0\})$ if there exists a neighborhood $\mathcal{U}_{\epsilon_0}(x_0) = \big\{x \in \mathbb{R}^n : |x - x_0| < \epsilon_0\big\}$ such that $\langle\xi\rangle^s\mathcal{F}(\psi f) \in L^2\big(\mathbb{R}^n\big)$ for all $\psi \in \mathcal{C}_0^\infty\big(\mathbb{R}^n\big)$ with $\operatorname{supp}\psi \subset \mathcal{U}_{\epsilon_0}(x_0)$. A function f belongs to the space $H^s_{\text{loc}}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ if $f\psi \in H^s\big(\mathbb{R}^n\big)$ for all $\psi \in \mathcal{C}_0^\infty(\Omega)$.
The proofs of the next two theorems follow the proof of Theorem 7.5 from the paper [88]. The main idea to prove the following theorems is to study the regularity properties of solutions for $\xi \in \mathcal{Z}_{\text{ext}}(\varepsilon)$. Specifically, we may rewrite the solution for large frequencies by

$$W^{(0)}(t,\xi) = \Big(\operatorname{diag}\big(\mathrm{e}^{-\mu_l(|\xi|)t}\big)_{l=1}^6 + T^{-1}_{\theta,\text{ext}}(|\xi|)\Big[\operatorname{diag}\big(\mathrm{e}^{-\mu_l(|\xi|)t}\big)_{l=1}^6, T_{\theta,\text{ext}}(|\xi|)\Big]\Big)W_0^{(0)}(\xi),$$

where the real parts of eigenvalues are nonnegative and have the properties

- when $\theta = 0$: Re $\mu_l(\xi) \lesssim 1/2$ for $l = 1,2,3,4,5,6$;
- when $\theta = 1$: Re $\mu_l(\xi) \lesssim b^2$ for $l = 1,2,3$, and Re $\mu_l(\xi) \gtrsim |\xi|^2$ for $l = 4,5,6$.

Then, according to the Fourier multiplier appearing in wave equations, one can complete the proof.

Theorem 2.3.2. *Let us consider the Cauchy problem* (2.1) *with* $\theta = 0$. *Assume that the initial data satisfies* $\nabla u_0^k, u_1^k \in H^s\big(\mathbb{R}^3\big)$ *but* $\nabla u_0^k, u_1^k \notin H^{s+1}_{\text{loc}}(x_1,x_2,x_3)$ *for* $k = 1,2,3$ *and a given point* $(x_1,x_2,x_3) \in \mathbb{R}^3$. *Then, the following statement holds:*

$$\nabla u^k(t,\cdot), u_t^k(t,\cdot) \notin H^{s+1}_{\text{loc}}(\{(x_1,x_2,x_3) \pm a\mathbf{e}t\} \cup \{(x_1,x_2,x_3) \pm b\mathbf{e}t\})$$

for all $t > 0$, *where* $\mathbf{e}$ *is an arbitrary unit vector in* $\mathbb{R}^3$.

Theorem 2.3.3. *Let us consider the Cauchy problem* (2.1) *with* $\theta = 1$. *Assume that the initial data satisfies* $\nabla u_0^k, u_1^k \in H^s(\mathbb{R}^3)$ *but* $\nabla u_0^k, u_1^k \notin H^{s+1}_{\text{loc}}(x_1, x_2, x_3)$ *for* $k = 1, 2, 3$ *and a given point* $(x_1, x_2, x_3) \in \mathbb{R}^3$. *Then, the following statement holds:*

$$\nabla u^k(t, \cdot), u_t^k(t, \cdot) \notin H^{s+1}_{\text{loc}}(\{(x_1, x_2, x_3) - a\mathbf{e}t\} \cup \{(x_1, x_2, x_3) - b\mathbf{e}t\})$$

for all $t > 0$, *where* $\mathbf{e}$ *is an arbitrary unit vector in* $\mathbb{R}^3$.

Remark 2.3.1. *Comparing the statements of Theorem 2.3.2 with Theorem 2.3.3, we observe that the propagation pictures are different taking account that the dominant parts of eigenvalues* $\mu_l(|\xi|)$ *in the cases* $l = 4, 5, 6$ *are equal to* $1/2$ *(for the case when* $\theta = 0$*) and* $|\xi|^2$ *(for the case when* $\theta = 1$*), respectively. It implies*

$$W_4^{(0)}, W_5^{(0)}, W_6^{(0)} \in \mathcal{C}\big([0, \infty), L^{2,s+1}(\mathbb{R}^3)\big),$$

that is,

$$\langle\xi\rangle^{s+1} W_4^{(0)}, \langle\xi\rangle^{s+1} W_5^{(0)}, \langle\xi\rangle^{s+1} W_6^{(0)} \in \mathcal{C}\big([0, \infty), L^2(\mathbb{R}^3)\big),$$

does not produce any singularities in these components in the model (2.1) *with viscoelastic damping.*

The problem of L^2 well-posedness can be immediately solved by the above results containing Gevrey smoothing if $\theta \in (0, 1)$ and propagation of singularities if $\theta = 0, 1$.

Theorem 2.3.4. *Let us consider the Cauchy problem* (2.1) *with* $\theta \in [0, 1]$ *and initial data satisfies* $\big(u_0^k, u_1^k\big) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *for* $k = 1, 2, 3$. *Then, there exists a uniquely determined energy solution*

$$u \in \big(\mathcal{C}\big([0, \infty), \dot{H}^1(\mathbb{R}^3)\big)\big)^3 \quad \text{and} \quad u_t \in \big(\mathcal{C}\big([0, \infty), L^2(\mathbb{R}^3)\big)\big)^3.$$

Proof. From Theorems 2.3.1 to 2.3.3, we may conclude $|D|u^k, u_t^k \in L^\infty\big([0, \infty), L^2(\mathbb{R}^3)\big)$. By the same approach proving continuity (in time) of energy solutions to the Cauchy problem for the classical wave equation, more careful considerations yield $|D|u^k, u_t^k \in \mathcal{C}\big([0, \infty), L^2(\mathbb{R}^3)\big)$. Thus, the proof is complete. □

2.4. Energy estimates

To begin with this section, we take the following notations for the function spaces.
For one thing, let us introduce for any $s \geqslant 0$ and $m \in [1, 2]$ the spaces

$$\mathcal{D}^s_{m,1}(\mathbb{R}^n) := \big(H^{s+1}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big) \times \big(H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big)$$

with the corresponding norm

$$\|(f, g)\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^n)} := \|f\|_{H^{s+1}(\mathbb{R}^n)} + \|f\|_{L^m(\mathbb{R}^n)} + \|g\|_{H^s(\mathbb{R}^n)} + \|g\|_{L^m(\mathbb{R}^n)}.$$

For another, we define for any $s \geqslant 0$ and $m \in [1, 2]$ the function space

$$\mathcal{D}^s_{m,2}(\mathbb{R}^n) := \big(|D|^{-1} H^s(\mathbb{R}^n) \cap \dot{H}^1_m(\mathbb{R}^n)\big) \times \big(H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big)$$

with the corresponding norm

$$\|(f, g)\|_{\mathcal{D}^s_{m,2}(\mathbb{R}^n)} := \|f\|_{|D|^{-1} H^s(\mathbb{R}^n)} + \|f\|_{\dot{H}^1_m(\mathbb{R}^n)} + \|g\|_{H^s(\mathbb{R}^n)} + \|g\|_{L^m(\mathbb{R}^n)}.$$

By $|D|^{-1} H^s(\mathbb{R}^n)$, for $s \in \mathbb{R}$, we denote the class of all distributions f from $\mathcal{Z}'(\mathbb{R}^n)$ such that

$$|D|^{-1} H^s(\mathbb{R}^n) := \big\{f \in \mathcal{Z}'(\mathbb{R}^n) : \|f\|_{|D|^{-1} H^s(\mathbb{R}^n)} := \||D| f\|_{H^s(\mathbb{R}^n)} < \infty\big\},$$

where $\mathcal{Z}'(\mathbb{R}^n)$ denotes the topological dual space to the subspace of Schwartz spaces $\mathcal{S}(\mathbb{R}^n)$ consisting of functions with $d_\xi^j \hat{f}(0) = 0$ for all $j \in \mathbb{N}$. In other words, we can identify $\mathcal{Z}'(\mathbb{R}^n)$ with the factor space $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$. Here $\mathcal{P}(\mathbb{R}^n)$ is the space of all polynomials [92]. We can discuss some properties of the distribution $f \in |D|^{-1}H^s(\mathbb{R}^n)$ in two ways. On the one hand, we may use some properties of the Bessel potential space $H^s(\mathbb{R}^n)$ because $|D|f \in H^s(\mathbb{R}^n)$. On the other hand, we may use some properties of the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^n)$ because $\langle D\rangle^s f \in \dot{H}^1(\mathbb{R}^n)$.

2.4.1. Energy estimates by using a diagonalization procedure

In this subsection we are going to derive the estimates for the classical energy and higher-order energies of solutions with initial data taking from $|D|^{-1}H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ with or without an additional regularity $\dot{H}^1_m(\mathbb{R}^3) \times L^m(\mathbb{R}^3)$, $m \in [1,2)$. The main tool of the proof is the use of asymptotic formulas of the solutions from Theorems 2.2.5 to 2.2.6. Since the proofs are quite standard, we only sketch them.

Theorem 2.4.1. *Let us consider the Cauchy problem* (2.1) *with* $\theta \in [0,1]$ *and initial data satisfies* $(u_0^k, u_1^k) \in \mathcal{D}^s_{2,2}(\mathbb{R}^3)$ *for* $k = 1,2,3$, *and* $s \geqslant 0$. *Then, we have the following estimates of energies of higher-order:*

$$\big\||D|^{s+1}u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} + \big\||D|^s u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{s}{2\max\{1-\theta;\theta\}}} \sum_{k=1}^{3} \big\|(u_0^k, u_1^k)\big\|_{\mathcal{D}^s_{2,2}(\mathbb{R}^3)}.$$

Proof. By virtue of the Parseval-Plancherel theorem and the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ for all $s \geqslant 0$ we can derive suitable estimates for the energies. □

Theorem 2.4.2. *Let us consider the Cauchy problem* (2.1) *with* $\theta \in [0,1]$ *and initial data satisfies* $(u_0^k, u_1^k) \in \dot{H}^{s+1}(\mathbb{R}^3) \times \dot{H}^s(\mathbb{R}^3)$ *for* $k = 1,2,3$, *and* $s \geqslant 0$. *Then, we have the following estimates for the energies of solutions of higher order:*

$$\big\||D|^{s+1}u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} + \big\||D|^s u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim \sum_{k=1}^{3} \big\|(u_0^k, u_1^k)\big\|_{\dot{H}^{s+1}(\mathbb{R}^3)\times\dot{H}^s(\mathbb{R}^3)}.$$

Proof. The estimates for the higher-order energies are determined by estimates for frequencies localized to the zone $\mathcal{Z}_{\text{int}}(\varepsilon)$. For this reason we may apply

$$\begin{aligned}
&\big\|\mathcal{F}^{-1}_{\xi\to x}\big(\chi_{\text{int}}(\xi)|\xi|^s T^{-1}_{\theta,\text{int}}(|\xi|)\,\text{diag}\left(\mathrm{e}^{-\mu_l(|\xi|)t}\right)_{l=1}^6 T_{\theta,\text{int}}(|\xi|)W_0^{(0)}(\xi)\big)\big\|_{(L^2(\mathbb{R}^3))^6}\\
&\lesssim \big\|\mathcal{F}^{-1}\big(|\xi|^s W_0^{(0)}(\xi)\big)\big\|_{(L^2(\mathbb{R}^3))^6}.
\end{aligned}$$

This yields our desired estimates. □

Now, we suppose an additional regularity $\dot{H}^1_m(\mathbb{R}^3) \times L^m(\mathbb{R}^3)$ for initial data with $m \in [1,2)$. This implies an additional decay in the corresponding estimates.

Theorem 2.4.3. *Let us consider the Cauchy problem* (2.1) *with* $\theta \in [0,1]$ *and initial data satisfies* $(u_0^k, u_1^k) \in \mathcal{D}^s_{m,2}(\mathbb{R}^3)$ *for* $k = 1,2,3$, *where* $s \geqslant 0$ *and* $m \in [1,2)$. *Then, we have the following estimates for the solutions and their energies of higher-order: if* $m \in [1,6/5)$, *then*

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{6-5m}{4m\max\{1-\theta;\theta\}}} \sum_{k=1}^{3} \big\|(u_0^k, u_1^k)\big\|_{\mathcal{D}^0_{m,2}(\mathbb{R}^3)},$$

and if $m \in [1,2)$, *then*

$$\big\||D|^{s+1}u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} + \big\||D|^s u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{3(2-m)+2ms}{4m\max\{1-\theta;\theta\}}} \sum_{k=1}^{3} \big\|(u_0^k, u_1^k)\big\|_{\mathcal{D}^s_{m,2}(\mathbb{R}^3)}.$$

Proof. For small frequencies, we apply Hölder's inequality and the Hausdorff-Young inequality to obtain the decay estimates

$$
\begin{aligned}
&\big\|\chi_{\text{int}}(D)|D|^{s+1}u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\big\|\chi_{\text{int}}(D)|D|^{s}u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\
&\lesssim \big\|\chi_{\text{int}}(\xi)|\xi|^{s}\mathrm{e}^{-c|\xi|^{2\max\{1-\theta;\theta\}}t}\big\|_{L^{\frac{2m}{2-m}}(\mathbb{R}^3)}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^0_{m,2}(\mathbb{R}^3)}\\
&\lesssim (1+t)^{-\frac{3(2-m)+2ms}{4m\max\{1-\theta;\theta\}}}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^0_{m,2}(\mathbb{R}^3)}.
\end{aligned}
$$

Moreover, an exponential decay estimate for the solutions with initial data belonging to $\mathcal{D}^s_{2,2}\big(\mathbb{R}^3\big)$ appears in the zone $\mathcal{Z}_{\text{ext}}(\varepsilon)$ of large frequencies.

However, when we discuss estimates for the solution itself by using Hölder's inequality, we immediately obtain

$$
\begin{aligned}
&\big\|\mathcal{F}^{-1}_{\xi\to x}\big(\chi_{\text{int}}(\xi)|\xi|^{-1}W^{(0)}(t,\xi)\big)\big\|_{(L^2(\mathbb{R}^3))^6}\\
&\lesssim \big\|\chi_{\text{int}}(\xi)|\xi|^{-1}\mathrm{e}^{-c|\xi|^{2\max\{1-\theta;\theta\}}t}\big\|_{L^{\frac{2m}{2-m}}(\mathbb{R}^3)}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\dot{H}^1_m(\mathbb{R}^3)\times L^m(\mathbb{R}^3)}.
\end{aligned}
$$

Therefore, we need to assume $m\in[1,6/5)$ to avoid a strong influence (non integrability) of the singularity as $|\xi|\to+0$, which means

$$
\big\|\chi_{\text{int}}(\xi)|\xi|^{-1}\mathrm{e}^{-c|\xi|^{2\max\{1-\theta;\theta\}}t}\big\|_{L^{\frac{2m}{2-m}}(\mathbb{R}^3)}<\infty\quad\text{for any}\quad t>0.
$$

So, the proof is complete. □

Remark 2.4.1. *If we use estimates for $\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}$ to derive estimates for $\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}$ with initial data belonging to $\mathcal{D}^0_{m,2}\big(\mathbb{R}^3\big)$ with $m\in[6/5,2]$, we need an additional assumption for initial data. To be more precise, we apply the following integral formula:*

$$
\int_0^t u_\tau^k(\tau,x)\mathrm{d}\tau=u^k(t,x)-u_0^k(x). \tag{2.26}
$$

Using estimates for $\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}$ we obtain for $\theta\in[0,1/2)\cup(1/2,1]$ the estimate

$$
\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim\big\|u_0^k\big\|_{L^2(\mathbb{R}^3)}+(1+t)^{1-\frac{3(2-m)}{4m\max\{1-\theta;\theta\}}}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^0_{m,2}(\mathbb{R}^3)}.
$$

Therefore, we suppose for initial data

$$
\big(u_0^k,u_1^k\big)\in\big(L^2\big(\mathbb{R}^3\big)\cap\dot{H}^1\big(\mathbb{R}^3\big)\cap\dot{H}^1_m\big(\mathbb{R}^3\big)\big)\times\big(L^2\big(\mathbb{R}^3\big)\cap L^m\big(\mathbb{R}^3\big)\big)
$$

for $m\in[6/5,2]$.

Remark 2.4.2. *We cannot expect energy estimates in Section 2.4.1 depending on a single initial data only due to the fact that from* (2.5) *and the diagonalization procedure, we obtain the representations of solutions by the coupling matrices $T_{\theta,\text{int}}(|\xi|)$ and $T_{\theta,\text{ext}}(|\xi|)$. These two matrices mix the influences of both data for estimating the solutions.*

Remark 2.4.3. *Concerning the sharpness of the derived estimates, we have to point out that the estimates of higher-order energies from Theorems 2.4.1 to 2.4.3 are sharp, where initial data is taken from*

$$
\big(|D|^{-1}H^s\big(\mathbb{R}^3\big)\cap\dot{H}^1_m\big(\mathbb{R}^3\big)\big)\times\big(H^s\big(\mathbb{R}^3\big)\cap L^m\big(\mathbb{R}^3\big)\big),
$$

for $s\geqslant 0$ and $m\in[1,2]$.

2.4.2. Energy estimates by using energy method in the Fourier space

To show the global (in time) existence of small data solutions, we find solutions in evolution spaces, this means, the solutions are continuous in time and H^{s+1}-valued with respect to the spatial variables. Therefore, we need Matsumura type (almost sharp) L^2 estimates for solutions to the linear elastic waves with different damping terms and initial data belonging to

$$\big(H^{s+1}(\mathbb{R}^3)\cap L^m(\mathbb{R}^3)\big)\times\big(H^{s}(\mathbb{R}^3)\cap L^m(\mathbb{R}^3)\big)$$

for $s\geqslant 0$ and $m\in[1,2]$.

We define the energy in the Fourier space $E_{\text{pha}}(\hat{u})=E_{\text{pha}}(\hat{u})(t,\xi)$ for all frequencies as follows:

$$E_{\text{pha}}(\hat{u}):=|\hat{u}_t|^2+a^2|\xi|^2|\hat{u}|^2+\big(b^2-a^2\big)|\xi\cdot\hat{u}|^2,$$

where dot $\cdot$ denotes the usual inner product in $\mathbb{R}^3$.

Lemma 2.4.1. *The energy $E_{\text{pha}}(\hat{u})$ of the Fourier image $\hat{u}=\hat{u}(t,\xi)$ of the solution $u=u(t,x)$ to the Cauchy problem* (2.1) *satisfies the following estimate:*

$$E_{\text{pha}}(\hat{u})(t,\xi)\lesssim\begin{cases}\mathrm{e}^{-c|\xi|^{2\max\{1-\theta;\theta\}}t}E_{\text{pha}}(\hat{u})(0,\xi) & \text{if } \xi\in\mathcal{Z}_{\text{int}}(\varepsilon),\\ \mathrm{e}^{-ct}E_{\text{pha}}(\hat{u})(0,\xi) & \text{if } \xi\in\mathcal{Z}_{\text{mid}}(\varepsilon)\cup\mathcal{Z}_{\text{ext}}(\varepsilon).\end{cases}$$

Proof. Applying the partial Fourier transformation to (2.1) we arrive at the following Cauchy problem in the Fourier space:

$$\begin{cases}\hat{u}_{tt}+a^2|\xi|^2\hat{u}+\big(b^2-a^2\big)(\xi\cdot\hat{u})\xi+|\xi|^{2\theta}\hat{u}_t=0, & \xi\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ (\hat{u},\hat{u}_t)(0,\xi)=(\hat{u}_0,\hat{u}_1)(\xi), & \xi\in\mathbb{R}^3.\end{cases}\tag{2.27}$$

Multiplying the equation that appears in the Cauchy problem (2.27) by the function $\bar{\hat{u}}_t$ we get

$$\frac{\partial}{\partial t}E_{\text{pha}}(\hat{u})+2|\xi|^{2\theta}|\hat{u}_t|^2=0.\tag{2.28}$$

To obtain decay rates for $E_{\text{pha}}(\hat{u})$ we divide the energy into two parts $\chi_{\text{int}}(\xi)E_{\text{pha}}(\hat{u})$ and $(1-\chi_{\text{int}}(\xi))E_{\text{pha}}(\hat{u})$ related to the zones $\mathcal{Z}_{\text{int}}(\varepsilon)$ and $\mathcal{Z}_{\text{mid}}(\varepsilon)\cup\mathcal{Z}_{\text{ext}}(\varepsilon)$, respectively.

For small frequencies, we multiply the function $|\xi|^\gamma\bar{\hat{u}}$ on both sides of the equations in the Cauchy problem (2.27) and take the real part to obtain

$$\begin{aligned}0=&\frac{\partial}{\partial t}\big(|\xi|^\gamma\text{Re}\big(\hat{u}_t\bar{\hat{u}}\big)\big)-|\xi|^\gamma|\hat{u}_t|^2+a^2|\xi|^{2+\gamma}|\hat{u}|^2\\&+\big(b^2-a^2\big)|\xi|^\gamma|\xi\cdot\hat{u}|^2+|\xi|^{2\theta+\gamma}\text{Re}\big(\hat{u}_t\bar{\hat{u}}\big),\end{aligned}\tag{2.29}$$

where the positive constant γ will be determined later. Adding (2.28) and (2.29) yields

$$\begin{aligned}\frac{\partial}{\partial t}\big(E_{\text{pha}}(\hat{u})+|\xi|^\gamma\text{Re}\big(\hat{u}_t\bar{\hat{u}}\big)\big)=&-\big(2|\xi|^{2\theta}-|\xi|^\gamma\big)|\hat{u}_t|^2-a^2|\xi|^{2+\gamma}|\hat{u}|^2\\&-\big(b^2-a^2\big)|\xi|^\gamma|\xi\cdot\hat{u}|^2-|\xi|^{2\theta+\gamma}\text{Re}\big(\hat{u}_t\bar{\hat{u}}\big).\end{aligned}\tag{2.30}$$

Combining (2.30) with Cauchy's inequality it follows

$$\begin{aligned}&\chi_{\text{int}}(\xi)\frac{\partial}{\partial t}\big(E_{\text{pha}}(\hat{u})+|\xi|^\gamma\text{Re}\big(\hat{u}_t\bar{\hat{u}}\big)\big)\\&\leqslant-\min\Big\{2|\xi|^{2\theta}-|\xi|^\gamma-\frac{|\xi|^{\tilde{\gamma}}}{4a^2};|\xi|^\gamma-|\xi|^{4\theta+2\gamma-\tilde{\gamma}-2}\Big\}\chi_{\text{int}}(\xi)E_{\text{pha}}(\hat{u}),\end{aligned}$$

where the positive constant $\tilde{\gamma}$ will be determined later. We state the restrictions for γ and $\tilde{\gamma}$, such that, $2\theta \leqslant \gamma$, $2\theta \leqslant \tilde{\gamma}$ and $\tilde{\gamma}+2-4\theta \leqslant \gamma$. These restrictions show that

$$\gamma \geqslant \max\{\tilde{\gamma}+2-4\theta; 2\theta\} \geqslant 2\max\{1-\theta;\theta\}.$$

Hence, we derive

$$\chi_{\text{int}}(\xi)\frac{\partial}{\partial t}\big(E_{\text{pha}}(\hat{u})+|\xi|^{\gamma}\text{Re}\big(\hat{u}_t\bar{\hat{u}}\big)\big) \leqslant -\chi_{\text{int}}(\xi)|\xi|^{\gamma}E_{\text{pha}}(\hat{u}).$$

Again, by virtue of Cauchy's inequality the energy term $\chi_{\text{int}}(\xi)E_{\text{pha}}(\hat{u})$ can be controlled as follows:

$$\chi_{\text{int}}(\xi)E_{\text{pha}}(\hat{u})(t,\xi) \leqslant 3\chi_{\text{int}}(\xi)\mathrm{e}^{-\frac{2}{3}|\xi|^{\gamma}t}E_{\text{pha}}(\hat{u})(0,\xi).$$

Hence, $\gamma = 2\max\{1-\theta;\theta\}$ and $\tilde{\gamma}=2\theta$ are the optimal choices, respectively.

For middle and large frequencies, we only multiply the equation which appears in the Cauchy problem (2.28) by the function $\bar{\hat{u}}$ and take the real part of it, i.e., $\gamma=0$ in (2.30), to get

$$(1-\chi_{\text{int}}(\xi))\frac{\partial}{\partial t}(E_{\text{pha}}(\hat{u})+\text{Re}\big(\hat{u}_t\bar{\hat{u}}\big)\big) \leqslant -\frac{1}{2}(1-\chi_{\text{int}}(\xi))E_{\text{pha}}(\hat{u})$$

due to $4\theta-2\leqslant 2\theta$ for all $\theta\in[0,1]$. Thus,

$$(1-\chi_{\text{int}}(\xi))\frac{\partial}{\partial t}\big(E_{\text{pha}}(\hat{u})+\text{Re}\big(\hat{u}_t\bar{\hat{u}}\big)\big) \leqslant -\frac{2}{3}(1-\chi_{\text{int}}(\xi))\big(E_{\text{pha}}(\hat{u})+\text{Re}\big(\hat{u}_t\bar{\hat{u}}\big)\big).$$

Finally, we may conclude

$$(1-\chi_{\text{int}}(\xi))E_{\text{pha}}(\hat{u})(t,\xi) \leqslant 3(1-\chi_{\text{int}}(\xi))\mathrm{e}^{-\frac{2}{3}t}E_{\text{pha}}(\hat{u})(0,\xi).$$

This completes the proof. $\square$

Theorem 2.4.4. *Let us consider the Cauchy problem* (2.1) *with* $\theta\in[0,1]$ *and initial data satisfies* $\big(u_0^k,u_1^k\big)\in\mathcal{D}_{2,1}^s\big(\mathbb{R}^3\big)$ *for* $k=1,2,3$, *and* $s\geqslant 0$. *Then, we have the following estimates:*

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}_{2,1}^0(\mathbb{R}^3)},$$

$$\big\||D|^{s+1}u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\big\||D|^{s}u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{s}{2\max\{1-\theta;\theta\}}}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}_{2,1}^s(\mathbb{R}^3)}.$$

Proof. By using the Parseval-Plancherel theorem the proof follows immediately from Lemma 2.4.1. $\square$

Now we turn to energy estimates with an additional regularity L^m, $m\in[1,2)$, for initial data. In the estimates for the solution there appears the time-dependent function $d_{0,m}=d_{0,m}(t)$ defined by

$$d_{0,m} := \begin{cases} (1+t)^{-\rho_0(m,\theta)} & \text{if } \theta\in\big[0,\frac{1}{2}\big),\quad m\in\big[1,\frac{6}{5}\big),\\ (1+t)^{1-\rho_1(m,\theta)} & \text{if } \theta\in\big[0,\frac{1}{2}\big),\quad m\in\big[\frac{6}{5},2\big),\\ (1+t)^{-\frac{6-5m}{4m\theta}} & \text{if } \theta\in\big[\frac{1}{2},1\big],\quad m\in\big[1,\frac{6}{5}\big),\\ (1+t)^{1-\frac{6-3m}{4m\theta}} & \text{if } \theta\in\big[\frac{1}{2},1\big],\quad m\in\big[\frac{6}{5},2\big), \end{cases}$$

where

$$\rho_0(m,\theta) < \min\Big\{\frac{6-5m+2m(1-2\theta)}{4m(1-\theta)};\frac{6-5m}{4m\theta}\Big\},$$

$$\rho_1(m,\theta) < \min\Big\{\frac{6-3m+2m(1-2\theta)}{4m(1-\theta)};\frac{6-3m}{4m\theta}\Big\}.$$

In the estimates for the energies of higher order of the solution there appears the time-dependent function $d_{s+1,m} = d_{s+1,m}(t)$ defined by

$$d_{s+1,m} := \begin{cases} (1+t)^{-\rho_{s+1}(m,\theta)} & \text{if } \theta \in \big[0, \frac{1}{2}\big), \\ (1+t)^{-\frac{6-3m+2sm}{4m\theta}} & \text{if } \theta \in \big[\frac{1}{2}, 1\big], \end{cases}$$

where $s \geqslant 0$, $m \in [1,2)$ and

$$\rho_{s+1}(m,\theta) < \min\Big\{\frac{6-3m+2sm+2m(1-2\theta)}{4m(1-\theta)}; \frac{6-3m+2sm}{4m\theta}\Big\}.$$

Theorem 2.4.5. *Let us consider the Cauchy problem* (2.1) *with* $\theta \in [0,1]$ *and initial data* $\big(u_0^k, u_1^k\big) \in \mathcal{D}^s_{m,1}\big(\mathbb{R}^3\big)$, *for* $k = 1,2,3$, $s \geqslant 0$, *and* $m \in [1,2)$. *Then, we have the following estimates:*

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim d_{0,m}(t) \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)},$$

$$\big\||D|^{s+1}u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} + \big\||D|^s u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim d_{s+1,m}(t) \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)}.$$

Proof. The estimates for the case $m = 1$ were studied in detail in [41]. Although the authors described the long-time behavior of the energy for $t \geqslant T\big(\|u_0\|_{H^1}, \|u_0\|_{L^1}, \|u_1\|_{L^2}, \|u_1\|_{L^1}\big)$, we know that a suitable energy of the solutions with initial data belonging to the space $\mathcal{D}^s_{m,1}\big(\mathbb{R}^3\big)$ is decaying for all $t \geqslant 0$ by the proof of Lemma 2.4.1.

First, we prove the estimates for the energies of higher order in the case when $m \in (1,2)$. Using the method from [41] if $\theta \in [0,1/2)$, then we introduce $d_{s+1,m}(t) = (1+t)^{-\rho_{s+1}(m,\theta)}$. Moreover, since we can follow the approach of the paper [41] we only need to prove

$$\int_{|\xi|\leqslant\varepsilon} |\xi|^{2s-4\theta\rho_{s+1}(m,\theta)} \big|\hat{u}_t^k(t,\xi)\big|^2 \mathrm{d}\xi \lesssim \sum_{k=1}^{3} \Big(\big\|u_0^k\big\|^2_{L^m(\mathbb{R}^3)} + \big\|u_1^k\big\|^2_{L^m(\mathbb{R}^3)}\Big),$$

$$\int_{|\xi|\leqslant\varepsilon} |\xi|^{2s-4(1-\theta)\rho_{s+1}(m,\theta)+2(1-2\theta)} \big|\hat{u}^k(t,\xi)\big|^2 \mathrm{d}\xi \lesssim \sum_{k=1}^{3} \Big(\big\|u_0^k\big\|^2_{L^m(\mathbb{R}^3)} + \big\|u_1^k\big\|^2_{L^m(\mathbb{R}^3)}\Big).$$

After applying Hölder's inequality and the Hausdorff-Young inequality, the above inequalities can be proved if we require

$$2(s-2\theta\rho_{s+1}(m,\theta))\frac{m}{2-m} + 2 > -1,$$
$$2(s-2(1-\theta)\rho_{s+1}(m,\theta)+1-2\theta)\frac{m}{2-m} + 2 > -1.$$

In conclusion, we assume

$$\rho_{s+1}(m,\theta) < \min\Big\{\frac{6-3m+2sm+2m(1-2\theta)}{4m(1-\theta)}; \frac{6-3m+2sm}{4m\theta}\Big\} \quad \text{if } \theta \in \big[0, \tfrac{1}{2}\big).$$

Following the same approach and setting $d_{0,m}(t) = (1+t)^{-\rho_0(m,\theta)}$, we also arrive at the estimate of the solution itself in the case when $m \in (1, 6/5)$. In the case when $\theta \in [1/2, 1]$, for the estimate of the solution itself with $m \in [1, 6/5)$ and for the estimate of the higher-order energies of the solutions with $m \in [1,2)$, after applying Lemma 2.4.1, Hölder's inequality and the Hausdorff-Young inequality we may conclude

$$\begin{aligned}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim & (1+t)^{-\frac{6-5m}{4m\theta}} \Big((1+t)^{-\frac{1}{2\theta}} \sum_{k=1}^{3} \big\|u_0^k\big\|_{L^m(\mathbb{R}^3)} + \sum_{k=1}^{3} \big\|u_1^k\big\|_{L^m(\mathbb{R}^3)}\Big) \\ & + \mathrm{e}^{-ct} \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)},\end{aligned}$$

$$\big\||D|^s u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{3(2-m)+2sm}{4m\theta}}\Big((1+t)^{-\frac{1}{2\theta}}\sum_{k=1}^{3}\big\|u_0^k\big\|_{L^m(\mathbb{R}^3)}+\sum_{k=1}^{3}\big\|u_1^k\big\|_{L^m(\mathbb{R}^3)}\Big)$$
$$+\,\mathrm{e}^{-ct}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{H^{s+1}(\mathbb{R}^3)\times H^s(\mathbb{R}^3)},$$

$$\big\||D|^{s+1} u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{3(2-m)+2sm}{4m\theta}}\Big((1+t)^{-\frac{1}{2\theta}}\sum_{k=1}^{3}\big\|u_0^k\big\|_{L^m(\mathbb{R}^3)}+\sum_{k=1}^{3}\big\|u_1^k\big\|_{L^m(\mathbb{R}^3)}\Big)$$
$$+\,\mathrm{e}^{-ct}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{H^{s+1}(\mathbb{R}^3)\times H^s(\mathbb{R}^3)}.$$

To get estimates for the solution itself with $m\in[6/5,2)$ for all $\theta\in[0,1]$, we can use the estimate for the $L^2\big(\mathbb{R}^3\big)$ norm of $u_t^k(t,\cdot)$ by applying the relation (2.26). Finally, let us mention that we could have better estimates by using the right-hand sides

$$\tilde{d}_1(t)\sum_{k=1}^{3}\big\|u_0^k\big\|+\tilde{d}_2(t)\sum_{k=1}^{3}\big\|u_1^k\big\|$$

with suitable norms. Nevertheless, our goal is to derive estimates with right-hand sides

$$\max\big\{\tilde{d}_1(t);\tilde{d}_2(t)\big\}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|$$

with suitable norms. □

Remark 2.4.4. *Moreover, when $\theta\in[0,1/2)$ we should point out that the total energy estimates from Theorem 2.4.5 with initial data being from $\big(H^{s+1}\big(\mathbb{R}^3\big)\cap L^1\big(\mathbb{R}^3\big)\big)\times\big(H^{s}\big(\mathbb{R}^3\big)\cap L^1\big(\mathbb{R}^3\big)\big)$ are almost sharp modulo a parameter $\epsilon>0$. Furthermore, the total energy estimates for the Cauchy problem* (2.1) *with structural damping $(-\Delta)^{1/2}u_t$ are sharp.*

Remark 2.4.5. *In the case when $\theta\in[1/2,1]$ with $m=1$, the decay rates in Theorem 2.4.5 are better than those of the paper [41] due to the fact that there is no any ambiguity of $\epsilon>0$ in the statements of Theorem 2.4.5.*

Theorem 2.4.6. *Let us consider the Cauchy problem* (2.1) *with $\theta\in[0,1]$ and initial data satisfies*

$$\big(u_0^k,u_1^k\big)\in\big(\dot{H}^{s+1}\big(\mathbb{R}^3\big)\cap L^m\big(\mathbb{R}^3\big)\big)\times\big(\dot{H}^{s}\big(\mathbb{R}^3\big)\cap L^m\big(\mathbb{R}^3\big)\big)$$

for $k=1,2,3$, $s\geqslant 0$, and $m\in[1,2)$. Then, we have the following estimates:

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim d_{0,m}(t)\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{(\dot{H}^1(\mathbb{R}^3)\cap L^m(\mathbb{R}^3))\times(L^2(\mathbb{R}^3)\cap L^m(\mathbb{R}^3))},$$
$$\big\||D|^s u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim d_{s+1,m}(t)\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{(\dot{H}^{s+1}(\mathbb{R}^3)\cap L^m(\mathbb{R}^3))\times(\dot{H}^s(\mathbb{R}^3)\cap L^m(\mathbb{R}^3))},$$
$$\big\||D|^{s+1} u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim d_{s+1,m}(t)\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{(\dot{H}^{s+1}(\mathbb{R}^3)\cap L^m(\mathbb{R}^3))\times(\dot{H}^s(\mathbb{R}^3)\cap L^m(\mathbb{R}^3))}.$$

Proof. The above assumptions for data allow modifying the considerations for large frequencies ξ. □

2.5. Diffusion phenomena

The diffusion phenomenon allows us to bridge the relation between a decay behavior of solutions to the dissipative elastic waves and a decay behavior of solutions to corresponding evolution systems with suitable initial data. According to Theorems 2.4.1 and 2.4.3 in the previous section, we know that energy estimates are determined by the behavior of the characteristic roots for small frequencies. For large frequencies, some regularity of initial data implies even an exponential decay. For this reason, the diffusion phenomenon is explained by the behavior of characteristic roots for small frequencies in this section.

To obtain a result on the diffusion phenomenon for our starting Cauchy problem (2.1), we choose the Cauchy problem for the following evolution reference systems with non-local operators from the structure and asymptotic behavior of characteristic roots for small frequencies:

$$\begin{cases}\widetilde{u}_t+\widetilde{M}_1(-\Delta)^{\sigma_1}\widetilde{u}+\widetilde{M}_2(-\Delta)^{\sigma_2}\widetilde{u}=0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ \widetilde{u}(0,x)=\widetilde{u}_0(x):=\mathcal{F}^{-1}\big(H(|\xi|)W_0^{(0)}(\xi)\big)(x), & x\in\mathbb{R}^3,\end{cases}\tag{2.31}$$

where the nonnegative constants σ_1,σ_2 and matrices $\widetilde{M}_1,\widetilde{M}_2$, $H(|\xi|)$ will be given later. It is clear that the solution to (2.31) can be represented by

$$\widetilde{u}(t,x)=\mathcal{F}^{-1}_{\xi\to x}\big(\operatorname{diag}\big(\mathrm{e}^{-\tilde{\mu}_l(|\xi|)t}\big)_{l=1}^{6}H(|\xi|)W_0^{(0)}(\xi)\big),\tag{2.32}$$

where the $\tilde{\mu}_l(|\xi|)$ are taking the principal values of the corresponding $\mu_l(|\xi|)$ from Theorem 2.2.5 and we will explain them in detail later. We denote $\widetilde{W}(t,\xi)=\mathcal{F}_{x\to\xi}(\widetilde{u}(t,x))$.

Remark 2.5.1. *Assume $\theta=1/2$ in the dissipative elastic waves* (2.1). *We observe that $\mathrm{e}^{-\frac{1}{2}|\xi|t}$ plays an important role in the representation of $W^{(1)}=W^{(1)}(t,\xi)$ from Case 2.3. Consequently, from direct calculations we conclude that there is not any improvement in the decay estimates for the difference between the solutions to the system* (2.1) *with $\theta=1/2$ and the solutions to its reference evolution system. For this reason, we only study diffusion phenomena to the dissipative system* (2.1) *with $\theta\in[0,1/2)\cup(1/2,1]$.*

2.5.1. Diffusion phenomenon for the linear model with $\theta=0$

According to the principal real part of $\mu_l(|\xi|)$ in Theorem 2.2.5, we choose $\sigma_1=1$, $\sigma_2=0$ and

$$\widetilde{M}_1=\operatorname{diag}\big(a^2,a^2,b^2,0,0,0\big)\quad\text{and}\quad\widetilde{M}_2=\operatorname{diag}(0,0,0,1,1,1)$$

in (2.31), that is,

$$\begin{cases}\widetilde{u}_t-\widetilde{M}_1\Delta\widetilde{u}+\widetilde{M}_2\widetilde{u}=0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ \widetilde{u}(0,x)=\widetilde{u}_0(x), & x\in\mathbb{R}^3.\end{cases}\tag{2.33}$$

Therefore, the eigenvalues in (2.32) are $\tilde{\mu}_{1,2}(|\xi|)=a^2|\xi|^2$, $\tilde{\mu}_3(|\xi|)=b^2|\xi|^2$ and $\tilde{\mu}_{4,5,6}(|\xi|)=1$, that is, we take the corresponding $\mu_l(|\xi|)$ from Theorem 2.2.5 after neglecting the terms $O\big(|\xi|^4\big)$ when $l=1,2,3$ and neglecting the terms $O\big(|\xi|^2\big)$ when $l=4,5,6$. Moreover, we define

$$H(|\xi|)=(I_{6\times6}+\mathcal{N}_3(|\xi|))^{-1}(I_{6\times6}+\mathcal{N}_2(|\xi|))^{-1}T_1^{-1},$$

where $T_1,\mathcal{N}_2(|\xi|),\mathcal{N}_3(|\xi|)$ have been defined in Subsection 2.2.2

Theorem 2.5.1. *Let us consider the Cauchy problem* (2.1) *with $\theta=0$. We assume that initial data satisfies $\big(u_0^k,u_1^k\big)\in\dot{H}^1_m\big(\mathbb{R}^3\big)\times L^m\big(\mathbb{R}^3\big)$ with $m\in[1,2]$ for $k=1,2,3$. Then, the following refinement estimates hold:*

$$\begin{aligned}&\big\|\chi_{\mathrm{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\big(W^{(0)}-T_1(I_{6\times6}+\mathcal{N}_2(|\xi|))(I_{6\times6}+\mathcal{N}_3(|\xi|))\widetilde{W}\big)(t,\cdot)\big\|_{(\dot{H}^s(\mathbb{R}^3))^6}\\&\lesssim(1+t)^{-\frac{3(2-m)+2ms}{4m}-\frac{1}{2}}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\dot{H}^1_m(\mathbb{R}^3)\times L^m(\mathbb{R}^3)},\end{aligned}$$

with $s \geqslant 0$.

Proof. We may rewrite the matrices by

$$T_{0,\mathrm{int}}(|\xi|) = H(|\xi|) + P_1(|\xi|), \quad T_{0,\mathrm{int}}^{-1}(|\xi|) = L(|\xi|) + P_2(|\xi|),$$

where

$$\begin{aligned} H(|\xi|) &= (I_{6\times 6} + \mathcal{N}_3(|\xi|))^{-1}(I_{6\times 6} + \mathcal{N}_2(|\xi|))^{-1}T_1^{-1}, \\ L(|\xi|) &= T_1(I_{6\times 6} + \mathcal{N}_2(|\xi|))(I_{6\times 6} + \mathcal{N}_3(|\xi|)), \end{aligned}$$

and $P_1(|\xi|), P_2(|\xi|) = O(|\xi|)$. Therefore, we decompose the function of interest into three parts, that is,

$$\chi_{\mathrm{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\big(W^{(0)} - \widetilde{W}\big)(t,x) = I_1(t,x) + I_2(t,x) + I_3(t,x),$$

where we denote

$$\begin{aligned} I_1(t,x) &= \mathcal{F}^{-1}_{\xi\to x}\Big(\chi_{\mathrm{int}}(\xi)L(|\xi|)\,\mathrm{diag}\left(\mathrm{e}^{-\mu_l(|\xi|)t} - \mathrm{e}^{-\tilde{\mu}_l(|\xi|)t}\right)_{l=1}^{6} H(|\xi|)W_0^{(0)}(\xi)\Big), \\ I_2(t,x) &= \mathcal{F}^{-1}_{\xi\to x}\Big(\chi_{\mathrm{int}}(\xi)L(|\xi|)\,\mathrm{diag}\left(\mathrm{e}^{-\mu_l(|\xi|)t}\right)_{l=1}^{6} P_1(|\xi|)W_0^{(0)}(\xi)\Big), \\ I_3(t,x) &= \mathcal{F}^{-1}_{\xi\to x}\Big(\chi_{\mathrm{int}}(\xi)P_2(|\xi|)\,\mathrm{diag}\left(\mathrm{e}^{-\mu_l(|\xi|)t}\right)_{l=1}^{6} T_{0,\mathrm{int}}(|\xi|)W_0^{(0)}(\xi)\Big). \end{aligned}$$

Here, we make use of the fact that

$$\mathrm{e}^{-\mu_l(|\xi|)t} - \mathrm{e}^{-\tilde{\mu}_l(|\xi|)t} = -r_l(|\xi|)t\mathrm{e}^{-\tilde{\mu}(|\xi|)t}\int_0^1 \mathrm{e}^{-r_l(|\xi|)t\tau}\mathrm{d}\tau$$

with $r_l = r_l(|\xi|)$ denoting the $O\big(|\xi|^4\big)$-terms from Theorem 2.2.5 for the corresponding $\mu_l(|\xi|)$. From energy estimates, we obtain

$$\|I_1(t,\cdot)\|_{(\dot{H}^s(\mathbb{R}^3))^6} \lesssim (1+t)^{-\frac{3(2-m)+2ms}{4m}-1}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\dot{H}^1_m(\mathbb{R}^3)\times L^m(\mathbb{R}^3)}.$$

Following the same procedure for the other two parts gives

$$\|I_2(t,\cdot)\|_{(\dot{H}^s(\mathbb{R}^3))^6} + \|I_3(t,\cdot)\|_{(\dot{H}^s(\mathbb{R}^3))^6} \lesssim (1+t)^{-\frac{3(2-m)+2ms}{4m}-\frac{1}{2}}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\dot{H}^1_m(\mathbb{R}^3)\times L^m(\mathbb{R}^3)}.$$

Summarizing above estimates we obtain

$$\begin{aligned} &\big\|\chi_{\mathrm{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\big(W^{(0)} - L(|\xi|)\widetilde{W}\big)(t,\cdot)\big\|_{(\dot{H}^s(\mathbb{R}^3))^6} \\ &\lesssim (1+t)^{-\frac{3(2-m)+2ms}{4m}-\frac{1}{2}}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\dot{H}^1_m(\mathbb{R}^3)\times L^m(\mathbb{R}^3)}. \end{aligned}$$

Thus, we complete the proof. □

2.5.2. Double diffusion phenomena for the linear model with $\theta \in (0,1/2)$

Similarly, according to the principal real part of $\mu_l(|\xi|)$ in Theorem 2.2.5 again, we assume $\sigma_1 = 1-\theta$, $\sigma_2 = \theta$ and

$$\widetilde{M}_1 = \mathrm{diag}\left(a^2,a^2,b^2,0,0,0\right) \quad \text{and} \quad \widetilde{M}_2 = \mathrm{diag}(0,0,0,1,1,1)$$

in (2.31), that is,

$$\begin{cases} \widetilde{u}_t + \widetilde{M}_1(-\Delta)^{1-\theta}\widetilde{u} + \widetilde{M}_2(-\Delta)^{\theta}\widetilde{u} = 0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+, \\ \widetilde{u}(0,x) = \widetilde{u}_0(x), & x\in\mathbb{R}^3. \end{cases} \tag{2.34}$$

Hence, we have $\tilde{\mu}_{1,2}(|\xi|)=a^2|\xi|^{2-2\theta}$, $\tilde{\mu}_3(|\xi|)=b^2|\xi|^{2-2\theta}$ and $\tilde{\mu}_{4,5,6}(|\xi|)=|\xi|^{2\theta}$. Moreover, we define

$$H(|\xi|)=(I_{6\times 6}+\mathcal{N}_3(|\xi|))^{-1}(I_{6\times 6}+\mathcal{N}_2(|\xi|))^{-1}T_1^{-1}$$

in (2.31) and (2.32), where $T_1,\mathcal{N}_2(|\xi|)$ and $\mathcal{N}_3(|\xi|)$ have been defined in Subsection 2.2.2.

Theorem 2.5.2. *Let us consider the Cauchy problem* (2.1) *with* $\theta\in(0,1/2)$. *We assume that initial data satisfies* $\big(u_0^k,u_1^k\big)\in\dot{H}^1_m\big(\mathbb{R}^3\big)\times L^m\big(\mathbb{R}^3\big)$ *with* $m\in[1,2]$ *for* $k=1,2,3$. *Then, the following refinement estimates hold:*

$$\begin{aligned}&\big\|\chi_{\mathrm{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\big(W^{(0)}-T_1(I_{6\times 6}+\mathcal{N}_2(|\xi|))(I_{6\times 6}+\mathcal{N}_3(|\xi|))\widetilde{W}\big)(t,\cdot)\big\|_{(\dot{H}^s(\mathbb{R}^3))^6}\\&\lesssim(1+t)^{-\frac{3(2-m)+2ms}{4m(1-\theta)}-\frac{1-2\theta}{2(1-\theta)}}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\dot{H}^1_m(\mathbb{R}^3)\times L^m(\mathbb{R}^3)},\end{aligned}$$

with $s\geqslant 0$.

Proof. Following the same steps of the proof of Theorem 2.5.1, we immediately conclude the statements of the theorem. $\square$

The asymptotic behavior of eigenvalues $\mu_{1,2,3}(|\xi|)=O\big(|\xi|^{2-2\theta}\big)$ and $\mu_{4,5,6}(|\xi|)=O\big(|\xi|^{2\theta}\big)$ in Theorem 2.2.5 is our motivation to study *double diffusion phenomena*. This new effect has been interpreted for the wave equation with structural damping $(-\Delta)^\theta u_t$ when $\theta\in(0,1/2)$ in the paper [21]. In fact, if we rewrite the solution to (2.34) as

$$\widetilde{u}(t,x)=\big(\widetilde{u}^+(t,x),\widetilde{u}^-(t,x)\big)^{\mathrm{T}},$$

the first part $\mathcal{F}^{-1}_{\xi\to x}\big(W^{(0)}_{1,2,3}\big)(t,x)$ behaves like the solution to the parabolic system with a suitable choice of initial data $\widetilde{u}_0^+(x)$, that is,

$$\begin{cases}\widetilde{u}_t^+ +\mathrm{diag}\,\big(a^2,a^2,b^2\big)(-\Delta)^{1-\theta}\widetilde{u}^+=0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ \widetilde{u}^+(0,x)=\widetilde{u}_0^+(x), & x\in\mathbb{R}^3.\end{cases}$$

The second part $\mathcal{F}^{-1}_{\xi\to x}\big(W^{(0)}_{4,5,6}\big)(t,x)$ behaves like the solution to another parabolic system with a suitable choice of initial data $\widetilde{u}_0^-(x)$, that is,

$$\begin{cases}\widetilde{u}_t^- +(-\Delta)^{\theta}\widetilde{u}^-=0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ \widetilde{u}^-(0,x)=\widetilde{u}_0^-(x), & x\in\mathbb{R}^3.\end{cases}$$

Nevertheless, due to the mixed influence from the matrices $T_{\theta,\mathrm{int}}(|\xi|)$ and $T^{-1}_{\theta,\mathrm{int}}(|\xi|)$, we only can observe the decay rate influenced by the eigenvalues $\mu_{1,2,3}(|\xi|)=O\big(|\xi|^{2-2\theta}\big)$ in Theorem 2.5.2.

2.5.3. Diffusion phenomenon for the linear model with $\theta\in(1/2,1]$

The components of $\mu_l(|\xi|)$ in Theorem 2.2.5 imply $\sigma_1=\theta$, $\sigma_2=1/2$ and

$$\widetilde{M_1}=\frac{1}{2}\,\mathrm{diag}(1,1,1,1,1,1)\quad\text{and}\quad\widetilde{M_2}=i\,\mathrm{diag}(a,a,b,-a,-a,-b)$$

in (2.31), that is,

$$\begin{cases}\widetilde{u}_t+\widetilde{M_1}(-\Delta)^{\theta}\widetilde{u}+\widetilde{M_2}(-\Delta)^{1/2}\widetilde{u}=0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ \widetilde{u}(0,x)=\widetilde{u}_0(x), & x\in\mathbb{R}^3.\end{cases}\tag{2.35}$$

So, the eigenvalues in (2.32) can be written as $\tilde{\mu}_{1,2}(|\xi|) = -i|\xi|a + \frac{1}{2}|\xi|^{2\theta}$, $\tilde{\mu}_3(|\xi|) = -i|\xi|b + \frac{1}{2}|\xi|^{2\theta}$, $\tilde{\mu}_{4,5}(|\xi|) = i|\xi|a + \frac{1}{2}|\xi|^{2\theta}$ and $\tilde{\mu}_6(|\xi|) = i|\xi|b + \frac{1}{2}|\xi|^{2\theta}$. We define

$$H(|\xi|) = (I_{6\times 6} + \mathcal{N}_2(|\xi|))^{-1}$$

in (2.31) and (2.32), where $\mathcal{N}_2(|\xi|)$ has been defined in Subsection 2.2.3.

Remark 2.5.2. *From the evolution system* (2.35)*, we observe that the reference system for the case when $\theta \in (1/2, 1]$ consists of a parabolic system and a half-wave system. This new effect has been interpreted for the wave equation with strong damping $-\Delta u_t$ in the papers [40, 47].*

Theorem 2.5.3. *Let us consider the Cauchy problem* (2.1) *with $\theta \in (1/2, 1]$. We assume that initial data satisfies $\big(u_0^k, u_1^k\big) \in \dot{H}^1_m\big(\mathbb{R}^3\big) \times L^m\big(\mathbb{R}^3\big)$ with $m \in [1, 2]$ for $k = 1, 2, 3$. Then, the following refinement estimates hold:*

$$\begin{aligned}
&\big\|\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\big(W^{(0)} - (I_{6\times 6} + \mathcal{N}_2(|\xi|))\widetilde{W}\big)(t,\cdot)\big\|_{(\dot{H}^s(\mathbb{R}^3))^6} \\
&\lesssim (1+t)^{-\frac{3(2-m)+2ms}{4m\theta} - \frac{2\theta-1}{2\theta}} \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{\dot{H}^1_m(\mathbb{R}^3)\times L^m(\mathbb{R}^3)},
\end{aligned}$$

with $s \geqslant 0$.

Proof. We follow the same steps of the proof of Theorem 2.5.1 to complete the result. □

Remark 2.5.3. *From Theorems 2.5.1 to 2.5.3, we found that the diffusion structure appears for linear elastic waves with friction or structural damping $(-\Delta)^\theta u_t$ if $\theta \in [0, 1/2) \cup (1/2, 1]$ as $t \to \infty$. That means if we compare the estimates from Theorems 2.5.1 to 2.5.3 with Theorems 2.4.1 and 2.4.3, we see the decay rate is improved by $-\frac{1-2\theta}{2(1-\theta)}$ when $\theta \in [0, 1/2)$ and $-\frac{2\theta-1}{2\theta}$ when $\theta \in (1/2, 1]$ as $t \to \infty$.*

2.6. Concluding remarks

Remark 2.6.1. *Sharp energy estimates for solutions to the Cauchy problem* (2.1) *with $\theta \in [0, 1/2) \cup (1/2, 1]$ and initial data*

$$\big(u_0^k, u_1^k\big) \in \big(H^{s+1}\big(\mathbb{R}^3\big) \cap L^m\big(\mathbb{R}^3\big)\big) \times \big(H^s\big(\mathbb{R}^3\big) \cap L^m\big(\mathbb{R}^3\big)\big)$$

for all $k = 1, 2, 3$ and $s \geqslant 0$, $m \in [1, 2)$, are still open.

Remark 2.6.2. *We can see from the theorems in this chapter that $\theta = 1/2$ is a critical value, as was pointed out in [62], in the sense that the dominant propagator is changed from $\theta \in [0, 1/2]$ to $\theta \in (1/2, 1]$. Furthermore, from the results about diffusion phenomena, we remark that the threshold of diffusion structure is $\theta = 1/2$ for elastic waves with structural damping. In other words, the structure of the corresponding reference system will be changed from $\theta \in (0, 1/2)$ to $\theta \in (1/2, 1]$.*

Remark 2.6.3. *From Section 2.5, we observe that for different choices of damping terms, which mainly depend on the value of the parameter θ in the Cauchy problem* (2.1)*, the diffusion phenomena are quite different. Precisely,*

- *in the case when $\theta = 0$, the reference system consists of a heat-type system;*
- *in the case when $\theta \in (0, 1/2)$, the reference system consists of two different parabolic systems;*
- *in the case when $\theta \in (1/2, 1]$, the reference system consists of a parabolic system and a half-wave system.*

Thus, it is interesting to study diffusion phenomena for the doubly dissipative elastic waves as follows:

$$\begin{cases} u_{tt}-a^2\Delta u-\big(b^2-a^2\big)\nabla\operatorname{div}u+(-\Delta)^{\rho}u_t+(-\Delta)^{\theta}u_t=0, & x\in\mathbb{R}^n,\ t\in\mathbb{R}_+,\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbb{R}^n, \end{cases}$$

where $b>a>0$, $n\geqslant 2$, and $0\leqslant\rho<1/2<\theta\leqslant 1$. The paper [8] gives an answer in the two dimensional case. The author obtained that $\rho+\theta=1$ is the threshold for diffusion structure. Precisely, in the case when $\rho+\theta<1$, the damping terms $(-\Delta)^{\rho}u_t$ and $(-\Delta)^{\theta}u_t$ with $0\leqslant\rho<1/2<\theta\leqslant 1$ have the influence on diffusion structure at the same time. However, in the case when $\rho+\theta\geqslant 1$, the diffusion structure is determined by the damping term $(-\Delta)^{\rho}u_t$ with $0\leqslant\rho<1/2$ only.

2.6.1. Summary

In this subsection we summarize some qualitative properties of solutions to linear elastic waves with friction or structural damping in 3D. Particularly, with the change of value of θ, the properties of solutions are changing. The main points can be summarised to be:

- Gevrey smoothing of solutions for $\theta\in(0,1)$, particularly, analytic smoothing for $\theta=1/2$;
- propagation of singularities for $\theta=0$ and $\theta=1$ with different propagation speeds;
- L^2 well-posedness for the Cauchy problem (2.1);
- energy estimates with initial data taking from the function spaces

$$\big(H^{s+1}(\mathbb{R}^3)\cap L^m(\mathbb{R}^3)\big)\times\big(H^s(\mathbb{R}^3)\cap L^m(\mathbb{R}^3)\big)$$

  or

$$\big(|D|^{-1}H^s(\mathbb{R}^3)\cap\dot{H}^1_m(\mathbb{R}^3)\big)\times\big(H^s(\mathbb{R}^3)\cap L^m(\mathbb{R}^3)\big)$$

  for $s\geqslant 0$ and $m\in[1,2]$;
- diffusion phenomena for $\theta\in[0,1/2)\cup(1/2,1]$, in particular, double diffusion phenomena for $\theta\in(0,1/2)$, basing on previous energy estimates.

2.6.2. Qualitative properties in a L^p-L^q framework

Throughout this chapter, we derive estimates of solutions in the L^2 norm. Nevertheless, concerning estimates of solutions in the L^q norm, where $2\leqslant q\leqslant\infty$, we may apply the following proposition.

Proposition 2.6.1. *Let us choose $f\in\mathcal{S}(\mathbb{R}^n)$ and $\kappa_1>0$, $\kappa_2\geqslant 0$, $s\geqslant 0$. Then, the next estimates hold:*

$$\big\|\mathcal{F}^{-1}_{\xi\to x}\big(\chi_{\text{int}}(\xi)|\xi|^s\mathrm{e}^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\big)\big\|_{L^q(\mathbb{R}^n)}\lesssim(1+t)^{-\frac{s}{\kappa_1}-\frac{n}{\kappa_1}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\mathbb{R}^n)}, \tag{2.36}$$

$$\big\|\mathcal{F}^{-1}_{\xi\to x}\big(\chi_{\text{ext}}(\xi)|\xi|^s\mathrm{e}^{-c|\xi|^{\kappa_2}t}\hat{f}(\xi)\big)\big\|_{L^q(\mathbb{R}^n)}\lesssim\mathrm{e}^{-ct}\big\|\langle D\rangle^{M_{n,s,p,q}}f\big\|_{L^p(\mathbb{R}^n)}, \tag{2.37}$$

where $c>0$, $1\leqslant p\leqslant 2\leqslant q\leqslant\infty$ and $M_{n,s,p,q}>s+n(1/p-1/q)$.

Remark 2.6.4. *If we are interested in the case $1<p\leqslant 2\leqslant q<\infty$, then we can choose the parameter in the regularity such that $M_{n,s,p,q}=s+n(1/p-1/q)$.*

Proof. Let us prove (2.36) first. Applying the Hausdorff-Young inequality yields

$$\big\|\mathcal{F}^{-1}_{\xi\to x}\big(\chi_{\text{int}}(\xi)|\xi|^s\mathrm{e}^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\big)\big\|_{L^q(\mathbb{R}^n)}\lesssim\big\|\chi_{\text{int}}(\xi)|\xi|^s\mathrm{e}^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\big\|_{L^{q'}(\mathbb{R}^n)}. \tag{2.38}$$

Here $1/q+1/q'=1$ with $2\leqslant q\leqslant\infty$. By Hölder's inequality, the following estimate holds:

$$\big\|\chi_{\rm int}(\xi)|\xi|^s\mathrm{e}^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\big\|_{L^{q'}(\mathbb{R}^n)}\lesssim\big\|\chi_{\rm int}(\xi)|\xi|^s\mathrm{e}^{-c|\xi|^{\kappa_1}t}\big\|_{L^{\tilde{p}}(\mathbb{R}^n)}\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)},\tag{2.39}$$

where $1/q'=1/\tilde{p}+1/p'$ with $2\leqslant p'\leqslant\infty$. Finally, combining with (2.38), (2.39) and the Hausdorff-Young inequality leads to

$$\big\|\mathcal{F}^{-1}_{\xi\to x}\big(\chi_{\rm int}(\xi)|\xi|^s\mathrm{e}^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\big)\big\|_{L^q(\mathbb{R}^n)}\lesssim(1+t)^{-\frac{s}{\kappa_1}-\frac{n}{\kappa_1}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\mathbb{R}^n)}.$$

Next, we begin with proving (2.37). For $0\leqslant t\leqslant 1$, by a similar approach we have

$$\begin{aligned}&\big\|\mathcal{F}^{-1}_{\xi\to x}\big(\chi_{\rm ext}(\xi)|\xi|^s\mathrm{e}^{-c|\xi|^{\kappa_2}t}\hat{f}(\xi)\big)\big\|_{L^q(\mathbb{R}^n)}\lesssim\big\|\chi_{\rm ext}(\xi)\langle\xi\rangle^s\hat{f}(\xi)\big\|_{L^{q'}(\mathbb{R}^n)}\\&\lesssim\Big\|\chi_{\rm ext}(\xi)\langle\xi\rangle^{-n\left(\frac{1}{p}-\frac{1}{q}\right)-\epsilon}\Big\|_{L^{\tilde{p}}(\mathbb{R}^n)}\Big\|\chi_{\rm ext}(\xi)\langle\xi\rangle^{s+n\left(\frac{1}{p}-\frac{1}{q}\right)+\epsilon}\hat{f}(\xi)\Big\|_{L^{p'}(\mathbb{R}^n)},\end{aligned}\tag{2.40}$$

where $1/q+1/q'=1$, $1/q'=1/\tilde{p}+1/p'$ with $2\leqslant q\leqslant\infty$ as well as $2\leqslant p'\leqslant\infty$ and arbitrary small constant $\epsilon>0$.
The following fact holds:

$$\Big\|\chi_{\rm ext}(\xi)\langle\xi\rangle^{-n\left(\frac{1}{p}-\frac{1}{q}\right)-\epsilon}\Big\|^{\tilde{p}}_{L^{\tilde{p}}(\mathbb{R}^n)}=\int_{1/\varepsilon}^{\infty}\langle r\rangle^{-n\left(\frac{1}{p}-\frac{1}{q}\right)\tilde{p}-\epsilon\tilde{p}+n-1}\mathrm{d}r=\int_{1/\varepsilon}^{\infty}\langle r\rangle^{-\epsilon\tilde{p}-1}\mathrm{d}r<\infty.\tag{2.41}$$

Summarizing (2.40), (2.41) and using the Hausdorff-Young inequality we derive

$$\big\|\mathcal{F}^{-1}_{\xi\to x}\big(\chi_{\rm ext}(\xi)|\xi|^s\mathrm{e}^{-c|\xi|^{\kappa_2}t}\hat{f}(\xi)\big)\big\|_{L^q(\mathbb{R}^n)}\lesssim\Big\|\langle D\rangle^{s+n\left(\frac{1}{p}-\frac{1}{q}\right)+\epsilon}f\Big\|_{L^p(\mathbb{R}^n)}$$

for $1\leqslant p\leqslant 2\leqslant q\leqslant\infty$ and $0\leqslant t\leqslant 1$.
For the case $t\geqslant 1$, according to $|\xi|\geqslant 1/\varepsilon$ we may obtain

$$\big\|\mathcal{F}^{-1}_{\xi\to x}\big(\chi_{\rm ext}(\xi)|\xi|^s\mathrm{e}^{-c|\xi|^{\kappa_2}t}\hat{f}(\xi)\big)\big\|_{L^q(\mathbb{R}^n)}\lesssim\mathrm{e}^{-ct}\Big\|\langle D\rangle^{s+n\left(\frac{1}{p}-\frac{1}{q}\right)+\epsilon}f\Big\|_{L^p(\mathbb{R}^n)}.$$

Hence, the proof of Proposition 2.6.1 is completed. $\square$

Thus, making use of Proposition 2.6.1, we may immediately obtain L^p-L^q estimates of solutions to the Cauchy problem (2.1), where $1\leqslant p\leqslant 2\leqslant q\leqslant\infty$. Furthermore, basing on these results, one may also derive results for diffusion phenomena in the L^p-L^q framework, which are similar as the statements of Theorems 2.5.1, 2.5.2 and 2.5.3.

3. Weakly coupled systems of the semilinear elastic waves with friction or structural damping in 3D

3.1. Introduction

This chapter is devoted to derive some global (in time) existence results for the following weakly coupled systems of elastic waves with friction or structural damping in 3D:

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + (-\Delta)^{\theta} u_t = f(u), & x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^3, \end{cases} \tag{3.1}$$

with $b > a > 0$ and $\theta \in [0, 1]$, where $u = \big(u^1, u^2, u^3\big)^{\mathrm{T}}$ and the nonlinear terms $f(u)$ on the right-hand sides can be represented by

$$f(u) := \big(|u^3|^{p_1}, |u^1|^{p_2}, |u^2|^{p_3}\big)^{\mathrm{T}}. \tag{3.2}$$

Here the exponents satisfy $p_1, p_2, p_3 > 1$.

Our main purpose of this chapter is to investigate the global (in time) existence of small data solutions to the Cauchy problem (3.1) with data belonging to function spaces of different regularity. The data spaces can be classified by classical energy spaces with suitable regularity, energy spaces with suitable higher regularity, and large regular Sobolev spaces. Thus, our first aim is to understand how does the space for initial data influence global (in time) existence results, particularly, the admissible range of the exponents p_1, p_2, p_3. Moreover, different nonlinear terms in (3.2) have different influences on the conditions for global (in time) existence results. We are interested in the interplay between the power nonlinearities to prove global (in time) existence of small data solutions. To do this, energy estimates for linear elastic waves with friction or structural damping (2.1), which have been proposed in the last chapter, play an important role. In order to estimate the nonlinear term in some norms, we may employ some inequalities in Harmonic Analysis, including the Gagliardo-Nirenberg inequalities, the fractional chain rule, the fractional Leibniz rule and the fractional powers rules (one may find them in Appendix B.2).

The rest of the chapter is organized as follows. In the remaining part of this section we introduce some exponents and parameters that will be used afterwards. In Section 3.2 we show the philosophy of our approach to prove global (in time) existence of small data solutions. In Sections 3.3 and 3.4 we prove global (in time) existence results for the Cauchy problem (3.1) with $\theta \in [0, 1/2)$ and $\theta \in [1/2, 1]$, respectively. In the last section some concluding remarks complete this chapter.

Preliminaries

Before stating our main results, we introduce for our further approach exponents $p_c(m, \theta)$, $\alpha_k(m, \theta)$ and $\widetilde{\alpha}_k(m, \theta)$ with some parameters $\theta \in [1/2, 1]$, $m \in [1, 6/5)$ and some balanced exponents.

Notation 1: We introduce the following exponent:

$$p_c(m, \theta) := 1 + \frac{m(2\theta + 1)}{3 - m} \quad \text{if} \quad \theta \in \big[\tfrac{1}{2}, 1\big] \quad \text{and} \quad m \in \big[1, \tfrac{6}{5}\big). \tag{3.3}$$

Remark 3.1.1. *As mentioned in the introduction of the thesis, in the case when $\theta \in [1/2, 1]$, the authors of [24] proved global (in time) existence of small data solutions to the following Cauchy* *problem:*

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^\theta u_t = |u|^p, & x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^3, \end{cases} \tag{3.4}$$

if we assume that the exponent satisfies $3/2 + \theta = p_c(1, \theta) < p \leqslant 3$. In other words, our exponents $p_c(1, \theta)$ for $\theta \in [1/2, 1]$ correspond to the global (in time) existence results for semilinear structurally damped wave equation in 3D. Especially, in the case when $\theta = 1/2$, the exponent $p_c\big(1, \frac{1}{2}\big) = 2$ corresponds to the critical exponent $p = 2$ for the Cauchy problem (3.4)*.*

Let us introduce the balanced exponents $p_{\text{bal}}\big(\frac{3}{2}, s, \theta\big)$ and $p_{\text{bal}}(m, 0, \theta)$, respectively,

$$p_{\text{bal}}\big(\tfrac{3}{2}, s, \theta\big) := 2 + \frac{2 + 4s(1-\theta)}{5 - 6\theta + 2s} \quad \text{if } \ m = \tfrac{3}{2},\ s \in \big[0, \tfrac{1}{2}\big),\ \theta \in \big[0, \tfrac{1}{2}\big),$$
$$p_{\text{bal}}(m, 0, \theta) := 2 + \frac{6(m - 2 + 2\theta)}{2m\theta - 3m + 6} \quad \text{if } \ m \in \big[\tfrac{6}{5}, \tfrac{3}{2}\big),\ s = 0,\ \theta \in \big[\tfrac{1}{2}, 1\big].$$

Notation 2: We define the following parameters:

$$\alpha_k(m, \theta) := m\Big(\frac{2\theta + (1+2\theta)p_{k+1} + p_k p_{k+1}}{2(p_k p_{k+1} - 1)}\Big) \ \text{ if } \ \theta \in \big[\tfrac{1}{2}, 1\big] \ \text{ and } \ m \in \big[1, \tfrac{6}{5}\big). \tag{3.5}$$

Remark 3.1.2. *The recent paper [19] proved that a global (in time) Sobolev solution to*

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^{1/2} u_t = |v|^{p_1}, & x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ v_{tt} - \Delta v + (-\Delta)^{1/2} v_t = |u|^{p_2}, & x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^3, \end{cases} \tag{3.6}$$

uniquely exists when the exponents fulfil

$$\alpha_{\max}\big(1, \tfrac{1}{2}\big) = \max\big\{\alpha_1\big(1, \tfrac{1}{2}\big); \alpha_2\big(1, \tfrac{1}{2}\big)\big\} = \frac{1 + 2\max\{p_1; p_2\} + p_1 p_2}{2(p_1 p_2 - 1)} < \frac{3}{2}.$$

Here we choose $p_1 = p_3$ in (3.5)*. It means that our parameters $\alpha_k\big(1, \frac{1}{2}\big)$ are related to the result for global (in time) existence of small data solutions to the Cauchy problem* (3.6)*.*

Remark 3.1.3. *We point out the relation between the parameters* (3.3) *and* (3.5)*. If we consider the condition $\alpha_k(m, \theta) < 3/2$, then this condition also can be rewritten as*

$$p_{k+1}(p_k + 1 - p_c(m, \theta)) > p_c(m, \theta),$$

where $\theta \in [1/2, 1]$ and $m \in [1, 6/5]$.

We introduce the balanced parameters $\alpha_{k,\text{bal}}\big(\frac{3}{2}, s, \theta\big)$ and $\alpha_{k,\text{bal}}(m, 0, \theta)$. If $s \in [0, 1/2)$ and $\theta \in [0, 1/2)$, then we introduce

$$\alpha_{k,\text{bal}}\big(\tfrac{3}{2}, s, \theta\big) := \frac{9 - 12\theta + 4s(2-\theta) + ((7 - 6\theta) + 2s(3 - 2\theta))p_{k+1} - ((2 - 6\theta) + 2s)p_k p_{k+1}}{2(p_k p_{k+1} - 1)}.$$

If $m \in [6/5, 3/2)$ and $\theta \in [1/2, 1]$, then we define

$$\alpha_{k,\text{bal}}(m, 0, \theta) := \frac{4m\theta + 12\theta - 3 + (2m\theta + 12\theta + 3m - 6)p_{k+1} - (2m\theta - 3m + 3)p_k p_{k+1}}{2(p_k p_{k+1} - 1)}.$$

Notation 3: We introduce the following parameter for $\theta \in [1/2, 1]$ and $m \in [1, 6/5)$:

$$\widetilde{\alpha}_k(m, \theta) := m\Big(\frac{2\theta + (1+2\theta)(p_{k+1} + 1)p_{k+2} + p_1 p_2 p_3}{2(p_1 p_2 p_3 - 1)}\Big). \tag{3.7}$$

Remark 3.1.4. *We now indicate a relation between the parameters* (3.3) *and* (3.7). *If we consider the condition* $\widetilde{\alpha}_k(m,\theta) < 3/2$*, then it is equivalent to*

$$p_{k+2}(p_{k+1}(p_k + 1 - p_c(m,\theta)) + 1 - p_c(m,\theta)) > p_c(m,\theta),$$

where $\theta \in [1/2,1]$ *and* $m \in [1,6/5)$.

Furthermore, the balanced parameters $\widetilde{\alpha}_{k,\text{bal}}\big(\frac{3}{2},s,\theta\big)$ and $\widetilde{\alpha}_{k,\text{bal}}(m,0,\theta)$ should be introduced. If $s \in [0,1/2)$ and $\theta \in [0,1/2)$, then we take the notation

$$\widetilde{\alpha}_{k,\text{bal}}\big(\tfrac{3}{2},s,\theta\big) := \frac{9-12\theta+4s(2-\theta)+((7-6\theta)+2s(3-2\theta))(p_{k+1}+1)p_{k+2}-((2-6\theta)+2s)p_1p_2p_3}{2(p_1p_2p_3-1)}.$$

If $m \in [6/5,3/2)$ and $\theta \in [1/2,1]$, then we denote

$$\widetilde{\alpha}_{k,\text{bal}}(m,0,\theta) := \frac{4m\theta+12\theta-3+(2m\theta+12\theta+3m-6)(p_{k+1}+1)p_{k+2}-(2m\theta-3m+3)p_1p_2p_3}{2(p_1p_2p_3-1)}.$$

Furthermore, we also introduce some notations to be used in this chapter.

Notation 4: For the sake of clarity, we denote k_1,k_2,k_3 as a triple which can choose the following three sets of numbers in this chapter:

- $k_1 = 1$, $k_2 = 2$ and $k_3 = 3$;
- $k_1 = 2$, $k_2 = 3$ and $k_3 = 1$;
- $k_1 = 3$, $k_2 = 1$ and $k_3 = 2$;

and re-define $p_{k_j+1} = p_{k_{j+1}}$, $p_{k_j+2} = p_{k_{j+2}}$ with $p_{k_4} = p_{k_1}$, $p_{k_5} = p_{k_2}$ for $j = 1,2,3$.
Moreover, we define $u^{k_j-1} = u^{k_{j-1}}$ with $u^{k_0} = u^{k_3}$ for $j = 1,2,3$.

Notation 5: Let us recall the function spaces $\mathcal{D}^s_{m,1}(\mathbb{R}^3)$ for any $s \geqslant 0$ and $m \in [1,2]$ such that

$$\mathcal{D}^s_{m,1}(\mathbb{R}^3) := \big(H^{s+1}(\mathbb{R}^3) \cap L^m(\mathbb{R}^3)\big) \times \big(H^s(\mathbb{R}^3) \cap L^m(\mathbb{R}^3)\big)$$

with the norm

$$\|(f,g)\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)} := \|f\|_{H^{s+1}(\mathbb{R}^3)} + \|f\|_{L^m(\mathbb{R}^3)} + \|g\|_{H^s(\mathbb{R}^3)} + \|g\|_{L^m(\mathbb{R}^3)}.$$

Finally, for the sake of self-containedness and readability of the chapter, we now recall some energy estimates for the linear Cauchy problem (2.1). The following result has been proved in the last chapter.

Theorem 3.1.1. *Let us consider the Cauchy problem* (2.1) *with* $\theta \in [0,1]$ *and initial data* $\big(u_0^k,u_1^k\big) \in \mathcal{D}^s_{m,1}(\mathbb{R}^3)$*, for* $k = 1,2,3$*,* $s \geqslant 0$*, and* $m \in [1,2)$*. Then, we have the following estimates:*

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim d_{0,m}(t)\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)},$$

$$\big\||D|^{s+1}u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} + \big\||D|^s u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim d_{s+1,m}(t)\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)}.$$

Here the time-dependent function $d_{0,m} = d_{0,m}(t)$ *is defined by*

$$d_{0,m} := \begin{cases} (1+t)^{-\rho_0(m,\theta)} & \text{if } \theta \in \left[0, \frac{1}{2}\right), \quad m \in \left[1, \frac{6}{5}\right), \\ (1+t)^{1-\rho_1(m,\theta)} & \text{if } \theta \in \left[0, \frac{1}{2}\right), \quad m \in \left[\frac{6}{5}, 2\right), \\ (1+t)^{-\frac{6-5m}{4m\theta}} & \text{if } \theta \in \left[\frac{1}{2}, 1\right], \quad m \in \left[1, \frac{6}{5}\right), \\ (1+t)^{1-\frac{6-3m}{4m\theta}} & \text{if } \theta \in \left[\frac{1}{2}, 1\right], \quad m \in \left[\frac{6}{5}, 2\right), \end{cases}$$

where

$$\rho_0(m,\theta) < \min\left\{\frac{6-5m+2m(1-2\theta)}{4m(1-\theta)}; \frac{6-5m}{4m\theta}\right\},$$
$$\rho_1(m,\theta) < \min\left\{\frac{6-3m+2m(1-2\theta)}{4m(1-\theta)}; \frac{6-3m}{4m\theta}\right\}.$$

Moreover, the time-dependent function $d_{s+1,m} = d_{s+1,m}(t)$ *is defined by*

$$d_{s+1,m} := \begin{cases} (1+t)^{-\rho_{s+1}(m,\theta)} & \text{if } \theta \in \left[0, \frac{1}{2}\right), \\ (1+t)^{-\frac{6-3m+2sm}{4m\theta}} & \text{if } \theta \in \left[\frac{1}{2}, 1\right], \end{cases}$$

where $s \geqslant 0$, $m \in [1,2)$ *and*

$$\rho_{s+1}(m,\theta) < \min\left\{\frac{6-3m+2sm+2m(1-2\theta)}{4m(1-\theta)}; \frac{6-3m+2sm}{4m\theta}\right\}.$$

3.2. Philosophy of our approach to study the global in time existence of solutions

In this section we explain our strategy to study the global (in time) existence of small data solutions to the Cauchy problem (3.1).

Let us consider the family of linear parameter dependent Cauchy problems

$$\begin{cases} u_{tt} - a^2\Delta u - \left(b^2 - a^2\right)\nabla \operatorname{div} u + (-\Delta)^\theta u_t = 0, & x \in \mathbb{R}^3, \ t > \tau, \\ (u, u_t)(\tau, x) = (u_0, u_1)(x), & x \in \mathbb{R}^3. \end{cases} \tag{3.8}$$

With the aim of studying the Cauchy problem (3.8), we define $K_0 = K_0(t,\tau,x)$, $K_1 = K_1(t,\tau,x)$ as the fundamental solutions with data $(u_0,u_1) = (\delta_0, 0)$ and $(u_0,u_1) = (0,\delta_0)$, respectively. Here δ_0 denotes the Dirac distribution in $x = 0$ with respect to the spatial variables. Then, the solution $u = u(t,x)$ to the Cauchy problem (3.8) is given by

$$u(t,x) = K_0(t,\tau,x) *_{(x)} u_0(x) + K_1(t,\tau,x) *_{(x)} u_1(x).$$

Next, by Duhamel's principle we see that

$$u(t,x) = \int_0^t K_1(t,\tau,x) *_{(x)} f(\tau,x) \mathrm{d}\tau$$

is the solution to the inhomogeneous linear Cauchy problem

$$\begin{cases} u_{tt} - a^2\Delta u - \left(b^2 - a^2\right)\nabla \operatorname{div} u + (-\Delta)^\theta u_t = f(t,x), & x \in \mathbb{R}^3, \ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (0, 0), & x \in \mathbb{R}^3. \end{cases}$$

We define on the family of complete spaces $\{X(T)\}_{T>0}$ the operator N as follows:

$$N : u \in X(T) \to Nu := (N_1 u, N_2 u, N_3 u)^{\mathrm{T}},$$

where for $k = 1, 2, 3$, we introduce

$$N_k u(t,x) = u^k_{\text{lin}}(t,x) + u^k_{\text{non}}(t,x)$$

with

$$\begin{aligned} u^k_{\text{lin}}(t,x) &:= K_0(t,0,x) *_{(x)} u^k_0(x) + K_1(t,0,x) *_{(x)} u^k_1(x), \\ u^k_{\text{non}}(t,x) &:= \int_0^t K_1(t,\tau,x) *_{(x)} |u^{k-1}(\tau,x)|^{p_k} \mathrm{d}\tau. \end{aligned}$$

The next inequalities play an essential role

$$\|Nu\|_{X(T)} \lesssim \sum_{k=1}^{3} \left\| \left(u^k_0, u^k_1\right) \right\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)} + \sum_{k=1}^{3} \|u\|^{p_k}_{X(T)}, \tag{3.9}$$

$$\|Nu - Nv\|_{X(T)} \lesssim \|u - v\|_{X(T)} \sum_{k=1}^{3} \left(\|u\|^{p_k - 1}_{X(T)} + \|v\|^{p_k - 1}_{X(T)} \right), \tag{3.10}$$

uniformly with respect to $T \in [0, \infty)$. They mainly show that the mapping $N : X(T) \to X(T)$ is a contraction for small data. Then, according to Banach's fixed-point theorem, there exists a uniquely determined solution $u^* = u^*(t,x)$ to the Cauchy problem (3.1) satisfying $Nu^* = u^* \in X(T)$ for all positive T. According to the estimates of solutions to the linearized Cauchy problem (c.f. Theorem 3.1.1) we have

$$\left\| \left(u^1_{\text{lin}}, u^2_{\text{lin}}, u^3_{\text{lin}}\right)^{\mathrm{T}} \right\|_{X(T)} \lesssim \sum_{k=1}^{3} \left\| \left(u^k_0, u^k_1\right) \right\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)}.$$

So, we complete the proof of all results separately by showing that

$$\left\| \left(u^1_{\text{non}}, u^2_{\text{non}}, u^3_{\text{non}}\right)^{\mathrm{T}} \right\|_{X(T)} \lesssim \sum_{k=1}^{3} \|u\|^{p_k}_{X(T)}. \tag{3.11}$$

The key tools to prove (3.11) and (3.10) are the Gagliardo-Nirenberg inequalities, the fractional chain rule, the fractional Leibniz rule and the fractional powers rules, which have been extensively and intensively discussed in Harmonic Analysis (e.g. Appendix B.2 or the book [25]).

Additionally, because different power source nonlinearities have different influences on conditions for the global (in time) existence of solutions, we allow the *effect of the loss of decay*, in particular, in the case that one of the exponents p_1, p_2, p_3 is below the exponent $p_c(m,\theta)$ or the balanced parameter $p_{\text{bal}}(m,s,\theta)$. For this reason we take the derived energy estimates for the solutions to the linear model (2.1) with vanishing right-hand sides and allow in the solution spaces some parameters describing a loss of decay.

We now state the strategy of the loss of decay. To prove the global (in time) existence of small data Sobolev solutions, the main difficulty is to estimate the integral over the interval $[0,t]$. We divide the interval $[0,t]$ in two sub-intervals $[0,t/2]$ and $[t/2,t]$. The difficulty is the estimate of the power nonlinearities in the norm of the solution space in each interval. If we allow to apply the Gagliardo-Nirenberg inequality, then there appear some relations including these parameters describing the loss of decay.

3.3. Existence results for semilinear elastic waves with friction or structural damping for $\theta \in [0, 1/2)$

From Theorem 3.1.1 we know that the time-dependent coefficients in the energy estimates for solutions to the Cauchy problem (2.1) depend continuously on the parameters $\theta \in [0, 1/2)$, $m \in [1, 2)$ and $s \geqslant 0$. In the following subsections we will choose the special cases when $m = 1$ or $m = 3/2$ to show clearly and succinctly our strategy to prove results for global (in time) existence of small data solutions.

3.3.1. Data from classical energy space with suitable regularity

Because initial data belongs to the function spaces $\mathcal{D}^0_{1,1}(\mathbb{R}^3)$ or $\mathcal{D}^0_{3/2,1}(\mathbb{R}^3)$ in this part, we mainly employ the classical Gagliardo-Nirenberg inequality to estimate nonlinearities in the L^2 norm and the L^m norm ($m=1$ or $m=3/2$). The restriction of admissible parameters from the application of the classical Gagliardo-Nirenberg inequality implies the condition $p_k\in[2/m,3]$ for all $k=1,2,3$.

For this reason, we may observe that in the following theorem all exponents are above the exponent $p=2$ (one may see Remark 3.3.1 for more detailed explanation).

Theorem 3.3.1. *Let us assume $p_k\in(2,3]$ for $k=1,2,3$. Then, there exists a constant $\varepsilon_0>0$ such that for initial data $(u_0^k,u_1^k)\in\mathcal{D}^0_{1,1}(\mathbb{R}^3)$ with*

$$\big\|(u_0^1,u_1^1)\big\|_{\mathcal{D}^0_{1,1}(\mathbb{R}^3)}+\big\|(u_0^2,u_1^2)\big\|_{\mathcal{D}^0_{1,1}(\mathbb{R}^3)}+\big\|(u_0^3,u_1^3)\big\|_{\mathcal{D}^0_{1,1}(\mathbb{R}^3)}\leqslant\varepsilon_0$$

there exists a uniquely determined energy solution

$$u\in\big(\mathcal{C}\big([0,\infty),H^1(\mathbb{R}^3)\big)\cap\mathcal{C}^1\big([0,\infty),L^2(\mathbb{R}^3)\big)\big)^3$$

to the Cauchy problem (3.1) *with $\theta\in[0,1/2)$. Moreover, the following estimates hold:*

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim(1+t)^{-\rho_0(1,\theta)}\sum_{k=1}^{3}\big\|(u_0^k,u_1^k)\big\|_{\mathcal{D}^0_{1,1}(\mathbb{R}^3)},$$

$$\big\|\nabla u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim(1+t)^{-\rho_1(1,\theta)}\sum_{k=1}^{3}\big\|(u_0^k,u_1^k)\big\|_{\mathcal{D}^0_{1,1}(\mathbb{R}^3)}.$$

Proof. Let us introduce the evolution space

$$X(T):=\big(\mathcal{C}\big([0,T],H^1(\mathbb{R}^3)\big)\cap\mathcal{C}^1\big([0,T],L^2(\mathbb{R}^3)\big)\big)^3 \tag{3.12}$$

with the corresponding norm

$$\begin{aligned}\|u\|_{X(T)}:=\sup_{t\in[0,T]}\Big(&\sum_{k=1}^{3}(1+t)^{\rho_0(1,\theta)}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\sum_{k=1}^{3}(1+t)^{\rho_1(1,\theta)}\big\|\nabla u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&+\sum_{k=1}^{3}(1+t)^{\rho_1(1,\theta)}\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\Big).\end{aligned}$$

In the definition of the norm the weights $(1+t)^{\rho_0(1,\theta)}$ and $(1+t)^{\rho_1(1,\theta)}$ come from the energy estimates for the corresponding linear Cauchy problem (2.1) with initial data belonging to $\mathcal{D}^0_{1,1}(\mathbb{R}^3)$.

Applying the classical Gagliardo-Nirenberg inequality we have

$$\begin{aligned}\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^m(\mathbb{R}^3)}&\lesssim\big\|u^{k-1}(\tau,\cdot)\big\|^{p_k(1-\beta_{0,1}(mp_k))}_{L^2(\mathbb{R}^3)}\big\|\nabla u^{k-1}(\tau,\cdot)\big\|^{p_k\beta_{0,1}(mp_k)}_{L^2(\mathbb{R}^3)}\\&\lesssim(1+\tau)^{-\rho_0(1,\theta)p_k+3\left(\frac{p_k}{2}-\frac{1}{m}\right)(\rho_0(1,\theta)-\rho_1(1,\theta))}\|u\|^{p_k}_{X(\tau)},\end{aligned}$$

where $\beta_{0,1}(mp_k)=3\big(\frac{1}{2}-\frac{1}{mp_k}\big)$ for $m=1$ and $m=2$. The restrictions from the application of the classical Gagliardo-Nirenberg inequality, i.e., $\beta_{0,1}(p_k),\beta_{0,1}(2p_k)\in[0,1]$, lead to $p_k\in[2,3]$ for all $k=1,2,3$.

Now we apply on $[0,t]$ the derived $\big(L^2\cap L^1\big)-L^2$ estimates for the solution itself to get

$$\begin{aligned}&(1+t)^{\rho_0(1,\theta)}\big\|u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&\lesssim(1+t)^{\rho_0(1,\theta)}\int_0^t(1+t-\tau)^{-\rho_0(1,\theta)}\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^2(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}\mathrm{d}\tau\\&\lesssim(1+t)^{\rho_0(1,\theta)}\|u\|^{p_k}_{X(t)}\int_0^t(1+t-\tau)^{-\rho_0(1,\theta)}(1+\tau)^{-\rho_0(1,\theta)p_k+3\left(\frac{p_k}{2}-1\right)(\rho_0(1,\theta)-\rho_1(1,\theta))}\mathrm{d}\tau,\end{aligned}$$

where we use $\|u\|_{X(\tau)} \leqslant \|u\|_{X(t)}$ for any $0 \leqslant \tau \leqslant t$. According to $(1+t-\tau) \approx (1+t)$ for any $\tau \in [0, t/2]$ and $(1+\tau) \approx (1+t)$ for any $\tau \in [t/2, t]$ we divide the interval $[0, t]$ into sub-intervals $[0, t/2]$ and $[t/2, t]$ to get

$$(1+t)^{\rho_0(1,\theta)} \big\|u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim \|u\|^{p_k}_{X(t)} \int_0^{t/2} (1+\tau)^{-\rho_0(1,\theta)p_k + 3\left(\frac{p_k}{2}-1\right)(\rho_0(1,\theta)-\rho_1(1,\theta))} \mathrm{d}\tau$$
$$+ (1+t)^{1-\rho_0(1,\theta)p_k+3\left(\frac{p_k}{2}-1\right)(\rho_0(1,\theta)-\rho_1(1,\theta))} \|u\|^{p_k}_{X(t)}.$$

Here we used $\rho_0(1,\theta) < 1$. Due to the assumption $p_k > 2$, we may use

$$p_k > 1 + \frac{2}{3-2\theta} \quad \text{with} \quad \theta \in \left[0, \tfrac{1}{2}\right),$$

for all $k = 1, 2, 3$. But then we have

$$-\rho_0(1,\theta)p_k + 3\Big(\frac{p_k}{2} - 1\Big)(\rho_0(1,\theta) - \rho_1(1,\theta)) < -1.$$

Therefore, it implies

$$(1+t)^{\rho_0(1,\theta)} \big\|u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim \|u\|^{p_k}_{X(t)}.$$

Similarly, we apply the derived $\left(L^2 \cap L^1\right) - L^2$ estimates on $[0, t/2]$ and the derived $L^2 - L^2$ estimates on $[t/2, t]$ to get

$$(1+t)^{\rho_1(1,\theta)} \big\|\partial_t^j \nabla^l u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim \|u\|^{p_k}_{X(t)} \int_0^{t/2} (1+\tau)^{-\rho_0(1,\theta)p_k+3\left(\frac{p_k}{2}-1\right)(\rho_0(1,\theta)-\rho_1(1,\theta))} \mathrm{d}\tau$$
$$+ (1+t)^{\rho_1(1,\theta)+1-\rho_0(1,\theta)p_k+\frac{3}{2}(p_k-1)(\rho_0(1,\theta)-\rho_1(1,\theta))} \|u\|^{p_k}_{X(t)}$$

for $j = 1$, $l = 0$ and $j = 0$, $l = 1$. Thanks to the condition $\min\{p_1; p_2; p_3\} > 2$, we have

$$\rho_1(1,\theta) + 1 - \rho_0(1,\theta)p_k + \frac{3}{2}(p_k - 1)(\rho_0(1,\theta) - \rho_1(1,\theta)) = \frac{3-2\theta}{2(1-\theta)}(2 - p_k) + \epsilon(p_k - 1) \leqslant 0,$$

where ϵ is a sufficiently small positive constant. The sufficient small constant $\epsilon > 0$ comes from the almost sharp energy estimates in Theorem 3.1.1, which can be written in the following way:

$$\big\|u^k_{\text{lin}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+\epsilon} \sum_{k=1}^{3} \big\|\left(u_0^k, u_1^k\right)\big\|_{\mathcal{D}^0_{1,1}(\mathbb{R}^3)},$$
$$\big\|\partial_t u^k_{\text{lin}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} + \big\|\nabla u^k_{\text{lin}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{5-4\theta}{4(1-\theta)}+\epsilon} \sum_{k=1}^{3} \big\|\left(u_0^k, u_1^k\right)\big\|_{\mathcal{D}^0_{1,1}(\mathbb{R}^3)}.$$

Here $u^k_{\text{lin}} = u^k_{\text{lin}}(t,x)$ is the solution to the Cauchy problem for linear elastic waves with friction or structural damping in 3D and with vanishing right-hand side.

Thus, the estimates for derivatives hold for all $k = 1, 2, 3$. In this way, we obtain for $j + l = 1$ and $j, l \in \mathbb{N}_0$

$$(1+t)^{\rho_1(1,\theta)} \big\|\partial_t^j \nabla^l u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim \|u\|^{p_k}_{X(t)}.$$

Next, we derive the Lipschitz condition by remarking that

$$\big\|\partial_t^j \nabla^l \left(u^k_{\text{non}} - v^k_{\text{non}}\right)(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}$$
$$= \Big\|\partial_t^j \nabla^l \int_0^t K_1(t-\tau, 0, x) *_{(x)} \left(|u^{k-1}(\tau,x)|^{p_k} - |v^{k-1}(\tau,x)|^{p_k}\right) \mathrm{d}\tau\Big\|_{L^2(\mathbb{R}^3)}.$$

Thanks to Hölder's inequality we get for $m=1,2$ the following estimates:

$$\begin{aligned}&\big\||u^{k-1}(\tau,\cdot)|^{p_k}-|v^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^m(\mathbb{R}^3)}\\&\lesssim \big\|u^{k-1}(\tau,\cdot)-v^{k-1}(\tau,\cdot)\big\|_{L^{mp_k}(\mathbb{R}^3)}\Big(\big\|u^{k-1}(\tau,\cdot)\big\|^{p_k-1}_{L^{mp_k}(\mathbb{R}^3)}+\big\|v^{k-1}(\tau,\cdot)\big\|^{p_k-1}_{L^{mp_k}(\mathbb{R}^3)}\Big).\end{aligned}$$

Similar as above we can use the classical Gagliardo-Nirenberg inequality again to estimate

$$\big\|u^{k-1}(\tau,\cdot)-v^{k-1}(\tau,\cdot)\big\|_{L^{mp_k}(\mathbb{R}^3)},\quad \big\|u^{k-1}(\tau,\cdot)\big\|_{L^{mp_k}(\mathbb{R}^3)},\quad \big\|v^{k-1}(\tau,\cdot)\big\|_{L^{mp_k}(\mathbb{R}^3)},$$

with $m=1,2$ and we can conclude (3.10). The proof is complete. □

Remark 3.3.1. *In Theorem 3.3.1, we allow that exponents p_1,p_2,p_3 are larger than the exponent $p=2$. If we assume that there exists a number $k_1=1,2,3$ such that $1<p_{k_1}<2$, the condition $p_{k_1}\in[2,3]$ from the application of the classical Gagliardo-Nirenberg inequality leads to an empty set for the exponent p_{k_1}.*

For initial data belonging to the classical energy spaces with an additional regularity $L^{3/2}$, we can obtain a larger admissible range of exponents p_1,p_2,p_3 because of the condition $p_k\in[4/3,3]$ for all $k=1,2,3$ coming from the application of the classical Gagliardo-Nirenberg inequality.

We observe the following three different cases.

Case 1: The orders of power nonlinearities are above the balanced exponent $p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$.
Case 2: Only one exponent is below or equal to the balanced exponent $p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$.
Case 3: Two exponents are below or equal to the balanced exponent $p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$.

Remark 3.3.2. *In Cases (ii) or (iii) of the next theorem, if some of the exponents $p_{k_j}=p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$, then we can choose the parameters g_{k_j} in the loss of decay as $g_{k_j}=\varepsilon_1$ with a sufficiently small constant $\varepsilon_1>0$ to avoid a logarithmic term $\log(e+t)$ in the estimate of the integral over $[0,t/2]$. Then, we can follow the proof of Theorem 3.3.2 without any new difficulties.*

Theorem 3.3.2. *Let us assume $p_k\in[4/3,3]$ for $k=1,2,3$, and the exponents satisfy one of the following conditions:*
(i) we assume

$$\min\{p_1;p_2;p_3\}>p_{\text{bal}}\big(\tfrac{3}{2},0,\theta\big); \tag{3.13}$$

(ii) we assume $\alpha_{k_1,\text{bal}}\big(\frac{3}{2},0,\theta\big)<3/2$ when

$$1<p_{k_1}<p_{\text{bal}}\big(\tfrac{3}{2},0,\theta\big)\quad\text{and}\quad p_{k_2},p_{k_3}>p_{\text{bal}}\big(\tfrac{3}{2},0,\theta\big); \tag{3.14}$$

(iii) we assume $\widetilde{\alpha}_{k_1,\text{bal}}\big(\frac{3}{2},0,\theta\big)<3/2$ when

$$1<p_{k_1},p_{k_2}<p_{\text{bal}}\big(\tfrac{3}{2},0,\theta\big)\quad\text{and}\quad p_{k_3}>p_{\text{bal}}\big(\tfrac{3}{2},0,\theta\big). \tag{3.15}$$

Then, there exists a constant $\varepsilon_0>0$ such that for initial data $\big(u_0^k,u_1^k\big)\in\mathcal{D}^0_{3/2,1}\big(\mathbb{R}^3\big)$ with

$$\big\|\big(u_0^1,u_1^1\big)\big\|_{\mathcal{D}^0_{3/2,1}(\mathbb{R}^3)}+\big\|\big(u_0^2,u_1^2\big)\big\|_{\mathcal{D}^0_{3/2,1}(\mathbb{R}^3)}+\big\|\big(u_0^3,u_1^3\big)\big\|_{\mathcal{D}^0_{3/2,1}(\mathbb{R}^3)}\leqslant\varepsilon_0$$

there is a uniquely determined energy solution

$$u\in\Big(\mathcal{C}\big([0,\infty),H^1(\mathbb{R}^3)\big)\cap\mathcal{C}^1\big([0,\infty),L^2(\mathbb{R}^3)\big)\Big)^3$$

to the Cauchy problem (3.1) with $\theta\in[0,1/2)$. Moreover, the following estimates hold:

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim(1+t)^{1-\rho_1(\frac{3}{2},\theta)+g_k}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^0_{3/2,1}(\mathbb{R}^3)},$$

$$\big\|\nabla u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim(1+t)^{-\rho_1(\frac{3}{2},\theta)+g_k}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^0_{3/2,1}(\mathbb{R}^3)},$$

where in the decay functions the numbers g_k are chosen in the following way:

1. $g_k = 0$ *for* $k = 1, 2, 3$, *when* p_1, p_2, p_3 *satisfy the condition* (3.13)*;*
2. $g_{k_1} = 3 + \big(\frac{1}{4(1-\theta)} - \frac{3}{2}\big)p_{k_1}$ *and* $g_{k_2} = g_{k_3} = 0$, *when* $p_{k_1}, p_{k_2}, p_{k_3}$ *satisfy the condition* (3.14)*;*
3. $g_{k_1} = 3 + \big(\frac{1}{4(1-\theta)} - \frac{3}{2}\big)p_{k_1}$, $g_{k_2} = 3 + \big(\frac{3}{2} + \frac{1}{4(1-\theta)}\big)p_{k_2} + \big(\frac{1}{4(1-\theta)} - \frac{3}{2}\big)p_{k_1}p_{k_2}$ *and* $g_{k_3} = 0$, *when* $p_{k_1}, p_{k_2}, p_{k_3}$ *satisfy the condition* (3.15)*.*

Proof. For any $T > 0$, let us introduce the evolution space (3.12) with the following norm:

$$\begin{aligned}\|u\|_{X(T)} := \sup_{t\in[0,T]}\Big(&\sum_{k=1}^{3}(1+t)^{-1+\rho_1(\frac{3}{2},\theta)-g_k}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&+\sum_{k=1}^{3}(1+t)^{\rho_1(\frac{3}{2},\theta)-g_k}\big(\big\|\nabla u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\big\|u^k_t(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\big)\Big).\end{aligned}\tag{3.16}$$

The classical Gagliardo-Nirenberg inequality implies

$$\begin{aligned}\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^{3/2}(\mathbb{R}^3)} &\lesssim (1+\tau)^{(1-\rho_1(\frac{3}{2},\theta))p_k-3(\frac{p_k}{2}-\frac{2}{3})+g_{k-1}p_k}\|u\|^{p_k}_{X(\tau)},\\ \big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^{2}(\mathbb{R}^3)} &\lesssim (1+\tau)^{(1-\rho_1(\frac{3}{2},\theta))p_k-3(\frac{p_k}{2}-\frac{1}{2})+g_{k-1}p_k}\|u\|^{p_k}_{X(\tau)}.\end{aligned}$$

The restriction of the parameters from applying the Gagliardo-Nirenberg inequality leads to $p_k \in [4/3, 3]$ for all $k = 1, 2, 3$.

Firstly, the application of the derived $\big(L^2 \cap L^{3/2}\big) - L^2$ estimate leads in the interval $[0, t]$ to

$$\begin{aligned}&(1+t)^{-1+\rho_1(\frac{3}{2},\theta)-g_k}\big\|u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&\lesssim (1+t)^{-1+\rho_1(\frac{3}{2},\theta)-g_k}\|u\|^{p_k}_{X(t)}\int_0^t(1+t-\tau)^{1-\rho_1(\frac{3}{2},\theta)}(1+\tau)^{(1-\rho_1(\frac{3}{2},\theta))p_k-3(\frac{p_k}{2}-\frac{2}{3})+g_{k-1}p_k}\mathrm{d}\tau.\end{aligned}$$

After dividing the interval $[0, t]$ into sub-intervals $[0, t/2]$ and $[t/2, t]$ it follows

$$\begin{aligned}&(1+t)^{-1+\rho_1(\frac{3}{2},\theta)-g_k}\big\|u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&\lesssim (1+t)^{-g_k}\|u\|^{p_k}_{X(t)}\int_0^{t/2}(1+\tau)^{(1-\rho_1(\frac{3}{2},\theta))p_k-3(\frac{p_k}{2}-\frac{2}{3})+g_{k-1}p_k}\mathrm{d}\tau\\&\quad+(1+t)^{1-g_k+(1-\rho_1(\frac{3}{2},\theta))p_k-3(\frac{p_k}{2}-\frac{2}{3})+g_{k-1}p_k}\|u\|^{p_k}_{X(t)},\end{aligned}$$

where we used the following estimate:

$$\int_{t/2}^{t}(1+t-\tau)^{1-\rho_1(\frac{3}{2},\theta)}\mathrm{d}\tau \lesssim (1+t)^{2-\rho_1(\frac{3}{2},\theta)} \quad \text{and} \quad \rho_1\big(\tfrac{3}{2},\theta\big) < 1.$$

In the same way, we may obtain the following estimates for the derivatives ($j + l = 1$):

$$\begin{aligned}&(1+t)^{\rho_1(\frac{3}{2},\theta)-g_k}\big\|\partial_t^j\nabla^l u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&\lesssim (1+t)^{-g_k}\|u\|^{p_k}_{X(t)}\int_0^{t/2}(1+\tau)^{(1-\rho_1(\frac{3}{2},\theta))p_k-3(\frac{p_k}{2}-\frac{2}{3})+g_{k-1}p_k}\mathrm{d}\tau\\&\quad+(1+t)^{1-g_k+(1-\rho_1(\frac{3}{2},\theta))p_k-3(\frac{p_k}{2}-\frac{2}{3})+g_{k-1}p_k}\|u\|^{p_k}_{X(t)}.\end{aligned}$$

Summarizing the above estimates we may conclude

$$\begin{aligned}&(1+t)^{(l+j)-1+\rho_1(\frac{3}{2},\theta)-g_k}\big\|\partial_t^j\nabla^l u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&\lesssim (1+t)^{-g_k}\|u\|^{p_k}_{X(t)}\int_0^{t/2}(1+\tau)^{2-(\frac{1}{2}+\rho_1(\frac{3}{2},\theta))p_k+g_{k-1}p_k}\mathrm{d}\tau\\&\quad+(1+t)^{3-(\frac{1}{2}+\rho_1(\frac{3}{2},\theta))p_k+g_{k-1}p_k-g_k}\|u\|^{p_k}_{X(t)},\end{aligned}\tag{3.17}$$

for all $j+l=0,1$ with $j,l\in\mathbb{N}_0$. In order to prove

$$(1+t)^{(l+j)-1+\rho_1(\frac{3}{2},\theta)-g_k}\big\|\partial_t^j\nabla^l u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim\|u\|^{p_k}_{X(t)},\tag{3.18}$$

we have to distinguish between three cases.

Case 1: We assume the condition (3.13), that is, $\min\{p_1;p_2;p_3\}>p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$.

Here the orders of power nonlinearities are above $p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$ and it allows to assume no loss of decay. Thus, we choose the parameters

$$g_1=g_2=g_3=0$$

and we get from the estimate (3.17)

$$\begin{aligned}&(1+t)^{(l+j)-1+\rho_1(\frac{3}{2},\theta)}\big\|\partial_t^j\nabla^l u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&\lesssim\|u\|^{p_k}_{X(t)}\Big(\int_0^{t/2}(1+\tau)^{2-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_k}\mathrm{d}\tau+(1+t)^{3-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_k}\Big),\end{aligned}$$

where $k=1,2,3$. If we guarantee

$$\min\{p_1;p_2;p_3\}>p_{\text{bal}}\big(\tfrac{3}{2},0,\theta\big)\quad\text{for}\quad\theta\in\big[0,\tfrac{1}{2}\big),$$

then we can prove

$$(1+\tau)^{2-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_k}\in L^1([0,\infty))\quad\text{and}\quad(1+t)^{3-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_k}\lesssim 1.$$

Thus, the desired estimate (3.18) holds for all $k=1,2,3$.

Case 2: We assume the condition (3.14), that is, $1<p_{k_1}<p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$ and $p_{k_2},p_{k_3}>p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$.

In this case, where only two exponents p_{k_2},p_{k_3} are above $p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$ we shall prove a global (in time) existence result with a loss of decay in one component of the solution under the additional condition

$$\alpha_{k_1,\text{bal}}\big(\tfrac{3}{2},0,\theta\big)=\frac{9-12\theta+(7-6\theta)p_{k_2}-(2-6\theta)p_{k_1}p_{k_2}}{2(p_{k_1}p_{k_2}-1)}<\frac{3}{2}.\tag{3.19}$$

We choose the parameters describing the loss of decay as

$$g_{k_1}=3+\Big(\frac{1}{4(1-\theta)}-\frac{3}{2}\Big)p_{k_1}\quad\text{and}\quad g_{k_2}=g_{k_3}=0.$$

The assumption $1<p_{k_1}<p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$ leads to $g_{k_1}>0$. Moreover, the condition (3.19) is equivalent to

$$12(1-\theta)+(7-6\theta)p_{k_2}-(5-6\theta)p_{k_1}p_{k_2}<0.\tag{3.20}$$

If we assume $p_{k_2}>p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$, then the condition (3.20) is valid.

Taking account (3.17) when $k=k_1$ and using the estimate

$$\int_0^{t/2}(1+\tau)^{2-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_1}}\mathrm{d}\tau\lesssim(1+t)^{3-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_1}}$$

because of $2-\big(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\big)p_{k_1}>-1$, we may conclude

$$(1+t)^{(l+j)-1+\rho_1(\frac{3}{2},\theta)-g_{k_1}}\big\|\partial_t^j\nabla^l u^{k_1}_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim(1+t)^{-g_{k_1}+3-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_1}}\|u\|^{p_{k_1}}_{X(t)}.\tag{3.21}$$

So, our desired estimate (3.18) has been proved for $k = k_1$.
Considering the case $k = k_2$, we obtain the following estimates:

$$\begin{aligned}
&(1+t)^{(l+j)-1+\rho_1(\frac{3}{2},\theta)}\big\|\partial_t^j\nabla^l u_{\text{non}}^{k_2}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\
&\lesssim \|u\|_{X(t)}^{p_{k_2}}\Big(\int_0^{t/2}(1+\tau)^{2-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_2}+g_{k_1}p_{k_2}}\mathrm{d}\tau+(1+t)^{3-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_2}+g_{k_1}p_{k_2}}\Big).
\end{aligned}$$

Taking account of (3.20) the following inequality holds:

$$2-\Big(\frac{1}{2}+\rho_1\big(\tfrac{3}{2},\theta\big)\Big)p_{k_2}+g_{k_1}p_{k_2}<-1.$$

Thus, it completes the estimate (3.18) for $k = k_2$.
Finally, we consider the case $k = k_3$. By the same procedure we treated *Case 1*, we immediately obtain

$$\begin{aligned}
&(1+t)^{(l+j)-1+\rho_1(\frac{3}{2},\theta)}\big\|\partial_t^j\nabla^l u_{\text{non}}^{k_3}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\
&\lesssim \|u\|_{X(t)}^{p_{k_3}}\Big(\int_0^{t/2}(1+\tau)^{2-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_3}}\mathrm{d}\tau+(1+t)^{3-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_3}}\Big)\lesssim\|u\|_{X(t)}^{p_{k_3}},
\end{aligned}$$

where we used our assumption $p_{k_3} > p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$.

Case 3: We assume the condition (3.15), that is, $1 < p_{k_1}, p_{k_2} < p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$ and $p_{k_3} > p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$.
Here there exists only one exponent p_{k_3} larger than $p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$. Hence, we shall prove a global (in time) existence of small data solutions result with a loss of decay for two components of the solution under the intersectional condition

$$\widetilde{\alpha}_{k_1,\text{bal}}\big(\tfrac{3}{2},0,\theta\big)=\frac{9-12\theta+(7-6\theta)(p_{k_2}+1)p_{k_3}-(2-6\theta)p_1p_2p_3}{2(p_1p_2p_3-1)}<\frac{3}{2}. \tag{3.22}$$

We choose the parameters as follows:

$$g_{k_1}=3+\Big(\frac{1}{4(1-\theta)}-\frac{3}{2}\Big)p_{k_1},\quad g_{k_2}=3+\Big(\frac{3}{2}+\frac{1}{4(1-\theta)}\Big)p_{k_2}+\Big(\frac{1}{4(1-\theta)}-\frac{3}{2}\Big)p_{k_1}p_{k_2}$$

and $g_{k_3} = 0$. With the help of the assumption $1 < p_{k_1}, p_{k_2} < p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$, we have $g_{k_1} > 0$ and $g_{k_2} > 0$. The condition (3.22) can be rewritten as

$$12(1-\theta)+(7-6\theta)(p_{k_2}+1)p_{k_3}-(5-6\theta)p_{k_1}p_{k_2}p_{k_3}<0. \tag{3.23}$$

If we assume $p_{k_3} > p_{\text{bal}}\big(\frac{3}{2},0,\theta\big)$, the above condition is valid.
When $k = k_1$ in the estimate (3.17), we can get the same estimates as (3.21) in *Case 2*. Choosing $k = k_2$, we apply the same method as we did in *Case 2* to get

$$\begin{aligned}
&(1+t)^{(l+j)-1+\rho_1(\frac{3}{2},\theta)-g_{k_2}}\big\|\partial_t^j\nabla^l u_{\text{non}}^{k_2}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\
&\lesssim (1+t)^{-g_{k_2}}\|u\|_{X(t)}^{p_{k_2}}\int_0^{t/2}(1+\tau)^{2-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_2}+g_{k_1}p_{k_2}}\mathrm{d}\tau\\
&\quad+(1+t)^{-g_{k_2}+3-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_2}+g_{k_1}p_{k_2}}\|u\|_{X(t)}^{p_{k_2}}\\
&\lesssim\|u\|_{X(t)}^{p_{k_2}},
\end{aligned}$$

where the choice of g_{k_2} leads to the inequality

$$g_{k_2}>3-\Big(\frac{1}{2}+\rho_1\big(\tfrac{3}{2},\theta\big)\Big)p_{k_2}+g_{k_1}p_{k_2}.$$

But this gives us for all $t \geqslant 0$ a uniformly bounded estimate from the above inequality.
Finally, let us take $k = k_3$ in the estimate (3.17). In the same way we get

$$(1+t)^{(l+j)-1+\rho_1(\frac{3}{2},\theta)} \big\|\partial_t^j \nabla^l u^{k_3}_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}$$
$$\lesssim \|u\|^{p_{k_3}}_{X(t)} \Big(\int_0^{t/2} (1+\tau)^{2-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_3}+g_{k_2}p_{k_3}} \mathrm{d}\tau + (1+t)^{3-\left(\frac{1}{2}+\rho_1(\frac{3}{2},\theta)\right)p_{k_3}+g_{k_2}p_{k_3}} \Big).$$

From the condition (3.23) it follows

$$2 - \Big(\frac{1}{2} + \rho_1\big(\tfrac{3}{2},\theta\big)\Big)p_{k_3} + g_{k_2}p_{k_3} < -1.$$

So, we immediately obtain our desired estimate (3.18) for $k = k_3$.
All in all, the estimate (3.18) has been completed for all the cases.

Lastly, similar as in the proof of Theorem 3.3.1, we may apply Hölder's inequality and the Gagliardo-Nirenberg inequality to prove

$$(1+t)^{(l+j)-1+\rho_1(\frac{3}{2},\theta)} \big\|\partial_t^j \nabla^l \big(u^k_{\text{non}} - v^k_{\text{non}}\big)(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim \|u-v\|_{X(t)} \big(\|u\|^{p_k-1}_{X(t)} + \|v\|^{p_k-1}_{X(t)}\big)$$

for all $j + l = 0, 1$ with $j, l \in \mathbb{N}_0$ and $k = 1, 2, 3$ in all cases. So, the proof is complete. □

3.3.2. Data from energy space with suitable higher regularity

In this subsection we are interested in studying global (in time) existence of small data energy solutions possessing energies of higher-order. As we know, the parameter $\min\{p_1; p_2; p_3\}$ is bounded to below by the regularity parameter $s + 1$.

Theorem 3.3.3. *Let us choose*

$$1 + \lceil s \rceil < \min\{p_1; p_2; p_3\} \leqslant \max\{p_1; p_2; p_3\} \leqslant 1 + \frac{2}{1-2s} \qquad \text{if } s \in \big(0, \tfrac{1}{2}\big),$$
$$1 + \lceil s \rceil < \min\{p_1; p_2; p_3\} \leqslant \max\{p_1; p_2; p_3\} < \infty \qquad \text{if } s \in \big[\tfrac{1}{2}, \infty\big).$$

Then, there exists a constant $\varepsilon_0 > 0$ *such that for initial data* $\big(u_0^k, u_1^k\big) \in \mathcal{D}^s_{1,1}\big(\mathbb{R}^3\big)$ *with*

$$\big\|\big(u_0^1, u_1^1\big)\big\|_{\mathcal{D}^s_{1,1}(\mathbb{R}^3)} + \big\|\big(u_0^2, u_1^2\big)\big\|_{\mathcal{D}^s_{1,1}(\mathbb{R}^3)} + \big\|\big(u_0^3, u_1^3\big)\big\|_{\mathcal{D}^s_{1,1}(\mathbb{R}^3)} \leqslant \varepsilon_0$$

there exists a uniquely determined energy solution

$$u \in \Big(\mathcal{C}\big([0,\infty), H^{s+1}\big(\mathbb{R}^3\big)\big) \cap \mathcal{C}^1\big([0,\infty), H^s\big(\mathbb{R}^3\big)\big)\Big)^3$$

to the Cauchy problem (3.1) *with* $\theta \in [0, 1/2)$. *Moreover, the following estimates hold:*

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\rho_0(1,\theta)} \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{\mathcal{D}^s_{1,1}(\mathbb{R}^3)},$$
$$\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\rho_1(1,\theta)} \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{\mathcal{D}^s_{1,1}(\mathbb{R}^3)},$$
$$\big\||D|u^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)} + \big\|u_t^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)} \lesssim (1+t)^{-\rho_{s+1}(1,\theta)} \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{\mathcal{D}^s_{1,1}(\mathbb{R}^3)}.$$

Remark 3.3.3. *In Theorem 3.3.3, our purpose is to weaken the upper bound for the exponents* p_1, p_2, p_3 *in comparison to the condition* $\max\{p_1; p_2; p_3\} \leqslant 3$ *in Theorem 3.3.1. After taking Bessel potential spaces with higher regularity for data, that is,* $\big(u_0^k, u_1^k\big) \in \mathcal{D}^s_{1,1}\big(\mathbb{R}^3\big)$ *with* $0 < s < 1/2$, *the largest admissible range for the exponents can be obtained when* $s - 1/2 \to -0$. *To improve the upper bound for* $\max\{p_1; p_2; p_3\}$, *we suppose more regularity for initial data.*

Proof. For any $T>0$ we define the complete evolution space

$$X(T) := \big(\mathcal{C}([0,T],H^{s+1}(\mathbb{R}^3)) \cap \mathcal{C}^1\big([0,T],H^s(\mathbb{R}^3)\big)\big)^3 \tag{3.24}$$

with the corresponding norm

$$\begin{aligned}\|u\|_{X(T)} = \sup_{t\in[0,T]}\Big(&\sum_{k=1}^{3}(1+t)^{\rho_0(1,\theta)}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} + \sum_{k=1}^{3}(1+t)^{\rho_1(1,\theta)}\big\|u^k_t(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\ &+\sum_{k=1}^{3}(1+t)^{\rho_{s+1}(1,\theta)}\big(\big\||D|u^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)} + \big\|u^k_t(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}\big)\Big).\end{aligned} \tag{3.25}$$

We shall estimate the norms $\big\|\partial_t^j u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}$, $\big\|\partial_t^j u^k_{\text{non}}(t,\cdot)\big\|_{\dot{H}^{s+1-j}(\mathbb{R}^3)}$ for $j=0,1$.
Firstly, using the derived $\big(L^2\cap L^1\big)-L^2$ estimates on the interval $[0,t]$ we have

$$\begin{aligned}&(1+t)^{\rho_0(1,\theta)}\big\|u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\ &\lesssim (1+t)^{\rho_0(1,\theta)}\int_0^t (1+t-\tau)^{-\rho_0(1,\theta)}\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^2(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}\mathrm{d}\tau.\end{aligned}$$

Next, the application of the $\big(L^2\cap L^1\big)-L^2$ estimates on $[0,t/2]$ and L^2-L^2 estimates on $[t/2,t]$ yields

$$\begin{aligned}(1+t)^{\rho_1(1,\theta)}\big\|\partial_t u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim &\int_0^{t/2}\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^2(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}\mathrm{d}\tau\\ &+(1+t)^{\rho_1(1,\theta)}\int_{t/2}^t\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^2(\mathbb{R}^3)}\mathrm{d}\tau.\end{aligned}$$

The fractional Gagliardo-Nirenberg inequality implies for $m=1,2$ the estimates

$$\begin{aligned}\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^m(\mathbb{R}^3)} &\lesssim \big\|u^{k-1}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^3)}^{(1-\beta_{0,s+1}(mp_k))p_k}\big\|u^{k-1}(\tau,\cdot)\big\|_{\dot{H}^{s+1}(\mathbb{R}^3)}^{\beta_{0,s+1}(mp_k)p_k}\\ &\lesssim (1+\tau)^{-\rho_0(1,\theta)p_k+\frac{3}{s+1}\left(\frac{p_k}{2}-\frac{1}{m}\right)(\rho_0(1,\theta)-\rho_{s+1}(1,\theta))}\|u\|_{X(\tau)}^{p_k},\end{aligned}$$

where

$$\beta_{0,s+1}(mp_k) = \frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{mp_k}\Big)\in[0,1].$$

This implies

$$\begin{aligned}&2\leqslant p_k\leqslant \frac{3}{1-2s} && \text{for } \ 0<s<\tfrac{1}{2}, \ \text{ or}\\ &2\leqslant p_k<\infty && \text{for } \ s\geqslant\tfrac{1}{2}.\end{aligned}$$

Hence, repeating the same procedure as in the proof of Theorem 3.3.1 and applying the assumption

$$\min\{p_1;p_2;p_3\}>2$$

we may conclude

$$(1+t)^{\rho_j(1,\theta)}\big\|\partial_t^j u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim\|u\|_{X(t)}^{p_k}$$

for $j=0,1$ and $k=1,2,3$.

Now, we estimate $u^k_{\text{non}}(t,\cdot)$ in the $\dot{H}^{s+1}$ norm and $\partial_t u^k_{\text{non}}(t,\cdot)$ in the $\dot{H}^s$ norm. The application of the derived $\big(\dot{H}^s\cap L^1\big)-\dot{H}^s$ estimates on $[0,t/2]$ and $\dot{H}^s-\dot{H}^s$ estimates on $[t/2,t]$ gives immediately

$$\begin{aligned}&(1+t)^{\rho_{s+1}(1,\theta)}\big\|\partial_t^j u^k_{\text{non}}(t,\cdot)\big\|_{\dot{H}^{s+1-j}(\mathbb{R}^3)}\\ &\lesssim\int_0^{t/2}\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}\mathrm{d}\tau+(1+t)^{\rho_{s+1}(1,\theta)}\int_{t/2}^t\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)}\mathrm{d}\tau.\end{aligned}$$

We should estimate the nonlinear term in the L^1 norm and the $\dot{H}^s$ norm, respectively. For the estimate of the L^1 norm, we can easily get from the fractional Gagliardo-Nirenberg inequality

$$\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^1(\mathbb{R}^3)} \lesssim (1+\tau)^{-\rho_0(1,\theta)p_k+\frac{3}{s+1}\left(\frac{p_k}{2}-1\right)(\rho_0(1,\theta)-\rho_{s+1}(1,\theta))}\|u\|^{p_k}_{X(\tau)}.$$

We apply more tools from Harmonic Analysis to estimate the $\dot{H}^s$ norm of $|u^{k-1}(\tau,\cdot)|^{p_k}$. Applying the fractional chain rule from Proposition B.2.4 we have

$$\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)} \lesssim \big\|u^{k-1}(\tau,\cdot)\big\|^{p_k-1}_{L^{q_1}(\mathbb{R}^3)}\big\|u^{k-1}(\tau,\cdot)\big\|_{\dot{H}^s_{q_2}(\mathbb{R}^3)}, \tag{3.26}$$

where $\frac{p_k-1}{q_1}+\frac{1}{q_2}=\frac{1}{2}$ and $p_k>\lceil s\rceil$ for $k=1,2,3$. Moreover, the fractional Gagliardo-Nirenberg-type inequality comes into play again to conclude

$$\begin{aligned}
\big\|u^{k-1}(\tau,\cdot)\big\|_{L^{q_1}(\mathbb{R}^3)} &\lesssim \big\|u^{k-1}(\tau,\cdot)\big\|^{1-\beta_{0,s+1}(q_1)}_{L^2(\mathbb{R}^3)}\big\|u^{k-1}(\tau,\cdot)\big\|^{\beta_{0,s+1}(q_1)}_{\dot{H}^{s+1}(\mathbb{R}^3)},\\
\big\|u^{k-1}(\tau,\cdot)\big\|_{\dot{H}^s_{q_2}(\mathbb{R}^3)} &\lesssim \big\|u^{k-1}(\tau,\cdot)\big\|^{1-\beta_{s,s+1}(q_2)}_{L^2(\mathbb{R}^3)}\big\|u^{k-1}(\tau,\cdot)\big\|^{\beta_{s,s+1}(q_2)}_{\dot{H}^{s+1}(\mathbb{R}^3)},
\end{aligned}$$

where

$$\beta_{0,s+1}(q_1)=\frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{q_1}\Big)\in[0,1] \quad\text{and}\quad \beta_{s,s+1}(q_2)=\frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{q_2}+\frac{s}{3}\Big)\in\Big[\frac{s}{s+1},1\Big].$$

The existence of parameters q_1 and q_2 will be discussed later. Combining with what we discussed above we arrive at the estimate

$$\begin{aligned}
&\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)}\\
&\lesssim \big\|u^{k-1}(\tau,\cdot)\big\|^{p_k-\frac{3}{s+1}\left(\frac{p_k-1}{2}+\frac{s}{3}\right)}_{L^2(\mathbb{R}^3)}\big\|u^{k-1}(\tau,\cdot)\big\|^{\frac{3}{s+1}\left(\frac{p_k-1}{2}+\frac{s}{3}\right)}_{\dot{H}^{s+1}(\mathbb{R}^3)}\\
&\lesssim (1+\tau)^{-\left(p_k-\frac{3}{s+1}\left(\frac{p_k-1}{2}+\frac{s}{3}\right)\right)\rho_0(1,\theta)-\frac{3}{s+1}\left(\frac{p_k-1}{2}+\frac{s}{3}\right)\rho_{s+1}(1,\theta)}\|u\|^{p_k}_{X(\tau)}.
\end{aligned} \tag{3.27}$$

Thus, by the condition $\min\{p_1;p_2;p_3\}>1+\lceil s\rceil\geqslant 2$ and the estimate

$$\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)} \lesssim (1+\tau)^{-\rho_0(1,\theta)p_k+\frac{3}{s+1}\left(\frac{p_k}{2}-1\right)(\rho_0(1,\theta)-\rho_{s+1}(1,\theta))}\|u\|^{p_k}_{X(\tau)},$$

we get

$$\begin{aligned}
&(1+t)^{\rho_{s+1}(1,\theta)}\big\|\partial_t^j u^k_{\text{non}}(t,\cdot)\big\|_{\dot{H}^{s+1-j}(\mathbb{R}^3)}\\
&\lesssim \|u\|^{p_k}_{X(t)}\int_0^{t/2}(1+\tau)^{-\rho_0(1,\theta)p_k+\frac{3}{s+1}\left(\frac{p_k}{2}-1\right)(\rho_0(1,\theta)-\rho_{s+1}(1,\theta))}d\tau\\
&\quad+(1+t)^{1+\rho_{s+1}(1,\theta)-\left(p_k-\frac{3}{s+1}\left(\frac{p_k-1}{2}+\frac{s}{3}\right)\right)\rho_0(1,\theta)-\frac{3}{s+1}\left(\frac{p_k-1}{2}+\frac{s}{3}\right)\rho_{s+1}(1,\theta)}\|u\|^{p_k}_{X(t)}\\
&\lesssim \|u\|^{p_k}_{X(t)}.
\end{aligned}$$

Summarizing all derived inequalities we get (3.9).

The last step is to derive the Lipschitz condition. The application of Hölder's inequality and the fractional Gagliardo-Nirenberg inequality yields for $j=0,1$

$$(1+t)^{\rho_j(1,\theta)}\big\|\partial_t^j\big(u^k_{\text{non}}-v^k_{\text{non}}\big)(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim \|u-v\|_{X(t)}\big(\|u\|^{p_k-1}_{X(t)}+\|v\|^{p_k-1}_{X(t)}\big).$$

In the following we will show how to estimate

$$\big\|\partial_t^j\big(u^k_{\text{non}}-v^k_{\text{non}}\big)(t,\cdot)\big\|_{\dot{H}^{s+1-j}(\mathbb{R}^3)} \quad\text{for}\quad j=0,1.$$

We apply the derived $\big(\dot{H}^s \cap L^1\big) - \dot{H}^s$ estimates and $\dot{H}^s - \dot{H}^s$ estimates again to conclude

$$
\begin{aligned}
&(1+t)^{\rho_{s+1}(1,\theta)} \big\|\partial_t^j \big(u^k_{\text{non}} - v^k_{\text{non}}\big)(t,\cdot)\big\|_{\dot{H}^{s+1-j}(\mathbb{R}^3)} \\
&\lesssim \int_0^{t/2} \big\||u^{k-1}(\tau,\cdot)|^{p_k} - |v^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)} \mathrm{d}\tau \\
&\quad + (1+t)^{\rho_{s+1}(1,\theta)} \int_{t/2}^t \big\||u^{k-1}(\tau,\cdot)|^{p_k} - |v^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)} \mathrm{d}\tau.
\end{aligned}
$$

To estimate the L^1 norm the fractional Gagliardo-Nirenberg inequality implies

$$
\begin{aligned}
&\big\||u^{k-1}(\tau,\cdot)|^{p_k} - |v^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^1(\mathbb{R}^3)} \\
&\lesssim (1+\tau)^{-\rho_0(1,\theta)p_k + \frac{3}{s+1}\left(\frac{p_k}{2}-1\right)(\rho_0(1,\theta)-\rho_{s+1}(1,\theta))} \|u-v\|_{X(\tau)} \big(\|u\|^{p_k-1}_{X(\tau)} + \|v\|^{p_k-1}_{X(\tau)}\big).
\end{aligned}
$$

We use the relation

$$
\frac{\mathrm{d}}{\mathrm{d}x_i}|x|^{p_k} = p_k|x|^{p_k-2}x_i
$$

and set

$$
g(u^{k-1}) = u^{k-1}|u^{k-1}|^{p_k-2}
$$

to get

$$
\begin{aligned}
&|u^{k-1}(\tau,x)|^{p_k} - |v^{k-1}(\tau,x)|^{p_k} \\
&= p_k \int_0^1 \big(u^{k-1}(\tau,x) - v^{k-1}(\tau,x)\big) g\big(\nu u^{k-1}(\tau,x) + (1-\nu)v^{k-1}(\tau,x)\big) \mathrm{d}\nu.
\end{aligned}
$$

Therefore, Minkowski's integral inequality and the fractional Leibniz rule from Proposition B.2.3 show that

$$
\begin{aligned}
&\big\||u^{k-1}(\tau,\cdot)|^{p_k} - |v^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)} \\
&\lesssim \int_0^1 \big\|\big(u^{k-1}(\tau,\cdot) - v^{k-1}(\tau,\cdot)\big) g\big(\nu u^{k-1}(\tau,\cdot) + (1-\nu)v^{k-1}(\tau,\cdot)\big)\big\|_{\dot{H}^s(\mathbb{R}^3)} \mathrm{d}\nu \\
&\lesssim \int_0^1 \big\|u^{k-1}(\tau,\cdot) - v^{k-1}(\tau,\cdot)\big\|_{\dot{H}^s_{r_1}(\mathbb{R}^3)} \big\|g\big(\nu u^{k-1}(\tau,\cdot) + (1-\nu)v^{k-1}(\tau,\cdot)\big)\big\|_{L^{r_2}(\mathbb{R}^3)} \mathrm{d}\nu \\
&\quad + \int_0^1 \big\|u^{k-1}(\tau,\cdot) - v^{k-1}(\tau,\cdot)\big\|_{L^{r_3}(\mathbb{R}^3)} \big\|g\big(\nu u^{k-1}(\tau,\cdot) + (1-\nu)v^{k-1}(\tau,\cdot)\big)\big\|_{\dot{H}^s_{r_4}(\mathbb{R}^3)} \mathrm{d}\nu,
\end{aligned}
$$

where $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4} = \frac{1}{2}$. Taking account of the first term on right-hand side we notice that

$$
\begin{aligned}
&\int_0^1 \big\|g\big(\nu u^{k-1}(\tau,\cdot) + (1-\nu)v^{k-1}(\tau,\cdot)\big)\big\|_{L^{r_2}(\mathbb{R}^3)} \mathrm{d}\nu \\
&\lesssim \big\|u^{k-1}(\tau,\cdot)\big\|^{p_k-1}_{L^{r_2(p_k-1)}(\mathbb{R}^3)} + \big\|v^{k-1}(\tau,\cdot)\big\|^{p_k-1}_{L^{r_2(p_k-1)}(\mathbb{R}^3)}.
\end{aligned}
$$

Actually, we use the fractional Gagliardo-Nirenberg inequality and the fractional chain rule to get the following inequalities:

$$
\begin{aligned}
&\big\|u^{k-1}(\tau,\cdot) - v^{k-1}(\tau,\cdot)\big\|_{\dot{H}^s_{r_1}(\mathbb{R}^3)} \\
&\lesssim \big\|u^{k-1}(\tau,\cdot) - v^{k-1}(\tau,\cdot)\big\|^{1-\beta_{s,s+1}(r_1)}_{L^2(\mathbb{R}^3)} \big\|u^{k-1}(\tau,\cdot) - v^{k-1}(\tau,\cdot)\big\|^{\beta_{s,s+1}(r_1)}_{\dot{H}^{s+1}(\mathbb{R}^3)}, \\
&\big\|u^{k-1}(\tau,\cdot) - v^{k-1}(\tau,\cdot)\big\|_{L^{r_3}(\mathbb{R}^3)} \\
&\lesssim \big\|u^{k-1}(\tau,\cdot) - v^{k-1}(\tau,\cdot)\big\|^{1-\beta_{0,s+1}(r_3)}_{L^2(\mathbb{R}^3)} \big\|u^{k-1}(\tau,\cdot) - v^{k-1}(\tau,\cdot)\big\|^{\beta_{0,s+1}(r_3)}_{\dot{H}^{s+1}(\mathbb{R}^3)},
\end{aligned}
$$

$$\begin{aligned}
\big\|u^{k-1}(\tau,\cdot)\big\|_{L^{r_2(p_k-1)}(\mathbb{R}^3)} &\lesssim \big\|u^{k-1}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^3)}^{1-\beta_{0,s+1}(r_2(p_k-1))}\big\|u^{k-1}(\tau,\cdot)\big\|_{\dot{H}^{s+1}(\mathbb{R}^3)}^{\beta_{0,s+1}(r_2(p_k-1))},\\
\big\|v^{k-1}(\tau,\cdot)\big\|_{L^{r_2(p_k-1)}(\mathbb{R}^3)} &\lesssim \big\|v^{k-1}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^3)}^{1-\beta_{0,s+1}(r_2(p_k-1))}\big\|v^{k-1}(\tau,\cdot)\big\|_{\dot{H}^{s+1}(\mathbb{R}^3)}^{\beta_{0,s+1}(r_2(p_k-1))},
\end{aligned}$$

and

$$\begin{aligned}
&\big\|g\big(\nu u^{k-1}(\tau,\cdot)+(1-\nu)v^{k-1}(\tau,\cdot)\big)\big\|_{\dot{H}^s_{r_4}(\mathbb{R}^3)}\\
&\lesssim \big\|\nu u^{k-1}(\tau,\cdot)+(1-\nu)v^{k-1}(\tau,\cdot)\big\|_{L^{r_5}(\mathbb{R}^3)}^{p_k-2}\big\|\nu u^{k-1}(\tau,\cdot)+(1-\nu)v^{k-1}(\tau,\cdot)\big\|_{\dot{H}^s_{r_6}(\mathbb{R}^3)}\\
&\lesssim \big(\big\|u^{k-1}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\big\|v^{k-1}(\tau,\cdot)\big\|_{L^2(\mathbb{R}^3)}\big)^{(1-\beta_{0,s+1}(r_5))(p_k-2)+1-\beta_{s,s+1}(r_6)}\\
&\quad\times \big(\big\|u^{k-1}(\tau,\cdot)\big\|_{\dot{H}^{s+1}(\mathbb{R}^3)}+\big\|v^{k-1}(\tau,\cdot)\big\|_{\dot{H}^{s+1}(\mathbb{R}^3)}\big)^{\beta_{0,s+1}(r_5)(p_k-2)+\beta_{s,s+1}(r_6)},
\end{aligned}$$

where the conditions for the parameters are

$$\begin{aligned}
\beta_{0,s+1}(r) &= \frac{3}{s+1}\Big(\frac12-\frac1r\Big)\in[0,1] &&\text{for } r=r_2(p_k-1),r_3,r_5,\\
\beta_{s,s+1}(r) &= \frac{3}{s+1}\Big(\frac12-\frac1r+\frac{s}{3}\Big)\in\Big[\frac{s}{s+1},1\Big] &&\text{for } r=r_1,r_6,\\
\frac{1}{r_4} &= \frac{p_k-2}{r_5}+\frac{1}{r_6},
\end{aligned}$$

for $\min\{p_1;p_2;p_3\}>\lceil s\rceil+1$. The existences of parameters $r_1,\dots,r_6$ are discussed in Section 3.5. Straightforward computations lead to

$$\begin{aligned}
\big\||u^{k-1}(\tau,\cdot)|^{p_k}-|v^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)} \lesssim &(1+\tau)^{-\big(p_k-\frac{3}{s+1}\big(\frac{p_k-1}{2}+\frac{s}{3}\big)\big)\rho_0(1,\theta)-\frac{3}{s+1}\big(\frac{p_k-1}{2}+\frac{s}{3}\big)\rho_{s+1}(1,\theta)}\\
&\times\|u-v\|_{X(\tau)}\big(\|u\|_{X(\tau)}^{p_k-1}+\|v\|_{X(\tau)}^{p_k-1}\big).
\end{aligned}$$

Summarizing all estimates allows to conclude (3.10). This completes the proof. □

Remark 3.3.4. *Again, in Theorem 3.3.3, we only expect that the exponents p_1,p_2,p_3 are above the exponent $p=2$. If we would assume $1<p_{k_1}\leqslant 2$, the admissible set for the exponent p_{k_1} will be empty. The reason is that to derive the Lipschitz condition, we apply the fractional chain rule and the fractional Leibniz rule. Therefore, we propose the condition* $\min\{p_1;p_2;p_3\}>1+\lceil s\rceil$.

By the same motivation for taking initial data from the space $\mathcal{D}^0_{3/2,1}(\mathbb{R}^3)$ we can obtain another admissible range for the exponents p_1,p_2,p_3 for proving the global (in time) existence of energy solutions with small data having an additional regularity $L^{3/2}$. Following the same approach as in the proofs to Theorems 3.3.2 and 3.3.3 we can prove the following result.

Theorem 3.3.4. *Let us assume $s\in(0,1/2)$. Let us choose*

$$1+\lceil s\rceil<\min\{p_1;p_2;p_3\}\leqslant\max\{p_1;p_2;p_3\}\leqslant 1+\frac{2}{1-2s}$$

and the exponents satisfy one of the following conditions:
(i) we assume

$$\min\{p_1;p_2;p_3\}>p_{\mathrm{bal}}\big(\tfrac32,s,\theta\big); \tag{3.28}$$

(ii) we assume $\alpha_{k_1,\mathrm{bal}}\big(\frac32,s,\theta\big)<3/2$ when

$$1<p_{k_1}<p_{\mathrm{bal}}\big(\tfrac32,s,\theta\big)\quad\text{and}\quad p_{k_2},p_{k_3}>p_{\mathrm{bal}}\big(\tfrac32,s,\theta\big); \tag{3.29}$$

(iii) we assume $\widetilde{\alpha}_{k_1,\mathrm{bal}}\big(\frac32,s,\theta\big)<3/2$ when

$$1<p_{k_1},p_{k_2}<p_{\mathrm{bal}}\big(\tfrac32,s,\theta\big)\quad\text{and}\quad p_{k_3}>p_{\mathrm{bal}}\big(\tfrac32,s,\theta\big). \tag{3.30}$$

Then, there exists a constant $\varepsilon_0 > 0$ *such that for initial data* $\big(u_0^k, u_1^k\big) \in \mathcal{D}^s_{3/2,1}\big(\mathbb{R}^3\big)$ *with*

$$\big\|\big(u_0^1, u_1^1\big)\big\|_{\mathcal{D}^s_{3/2,1}(\mathbb{R}^3)} + \big\|\big(u_0^2, u_1^2\big)\big\|_{\mathcal{D}^s_{3/2,1}(\mathbb{R}^3)} + \big\|\big(u_0^3, u_1^3\big)\big\|_{\mathcal{D}^s_{3/2,1}(\mathbb{R}^3)} \leqslant \varepsilon_0$$

there exists a uniquely determined energy solution

$$u \in \Big(\mathcal{C}\big([0,\infty), H^{s+1}\big(\mathbb{R}^3\big)\big) \cap \mathcal{C}^1\big([0,\infty), H^s\big(\mathbb{R}^3\big)\big)\Big)^3$$

to the Cauchy problem (3.1) *with* $\theta \in [0,1/2)$. *Moreover, the following estimates hold:*

$$\begin{aligned}
\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} &\lesssim (1+t)^{1-\rho_1(3/2,\theta)+g_k} \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{\mathcal{D}^s_{3/2,1}(\mathbb{R}^3)},\\
\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} &\lesssim (1+t)^{-\rho_1(3/2,\theta)+g_k} \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{\mathcal{D}^s_{3/2,1}(\mathbb{R}^3)},\\
\big\||D|u^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)} + \big\|u_t^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)} &\lesssim (1+t)^{-\rho_{s+1}(3/2,\theta)+g_k} \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{\mathcal{D}^s_{3/2,1}(\mathbb{R}^3)},
\end{aligned}$$

where the parameters g_k *are chosen in the following way:*

1. $g_k = 0$ *for* $k = 1,2,3$, *when* p_1, p_2, p_3 *satisfy the condition* (3.28)*;*
2. $g_{k_1} = 1 + \frac{2-2\theta+s}{(1-\theta)(s+1)} + \frac{6\theta-5-2s}{4(1-\theta)(s+1)}p_{k_1}$ *and* $g_{k_2} = g_{k_3} = 0$, *when* $p_{k_1}, p_{k_2}, p_{k_3}$ *satisfy the condition* (3.29)*;*
3. $g_{k_1} = 1 + \frac{2-2\theta+s}{(1-\theta)(s+1)} + \frac{6\theta-5-2s}{4(1-\theta)(s+1)}p_{k_1}$, $g_{k_2} = 1 + \frac{2-2\theta+s}{(1-\theta)(s+1)} + \big(1 + \frac{3+2s-2\theta}{4(1-\theta)(s+1)}\big)p_{k_2} + \frac{6\theta-5-2s}{4(1-\theta)(s+1)}p_{k_1}p_{k_2}$, *and* $g_{k_3} = 0$ *when* $p_{k_1}, p_{k_2}, p_{k_3}$ *satisfy the condition* (3.30).

Remark 3.3.5. *As stated in Remark 3.3.2, if some of the exponents* $p_{k_j} = p_{\text{bal}}\big(\frac{3}{2}, s, \theta\big)$ *for some* $j = 1,2,3$ *in Cases (ii) or (iii) in Theorem 3.3.4, then we can choose the parameters* g_{k_j} *describing the loss of decay as* $g_{k_j} = \varepsilon_1$ *with a sufficiently small constant* $\varepsilon_1 > 0$.

To end this section, let us say a few things about the case $s > 3/2$, where we suppose that initial data belongs to $\mathcal{D}^s_{1,1}\big(\mathbb{R}^3\big)$. Applying the fractional chain rule from Proposition B.2.4 would lead to the admissible range for the exponents p_1, p_2, p_3 such that

$$\min\{p_1; p_2; p_3\} > 1 + \lceil s \rceil.$$

Applying instead fractional powers rules from Proposition B.2.5 the last condition of the exponents can be relaxed to

$$\min\{p_1; p_2; p_3\} > 1 + s.$$

This is explained in the next result.

Theorem 3.3.5. *Let us assume* $s > 3/2$. *Let us choose*

$$1 + s < \min\{p_1; p_2; p_3\}.$$

Then, there exists a constant $\varepsilon_0 > 0$ *such that for all initial data* $\big(u_0^k, u_1^k\big) \in \mathcal{D}^s_{1,1}\big(\mathbb{R}^3\big)$ *with*

$$\big\|\big(u_0^1, u_1^1\big)\big\|_{\mathcal{D}^s_{1,1}(\mathbb{R}^3)} + \big\|\big(u_0^2, u_1^2\big)\big\|_{\mathcal{D}^s_{1,1}(\mathbb{R}^3)} + \big\|\big(u_0^3, u_1^3\big)\big\|_{\mathcal{D}^s_{1,1}(\mathbb{R}^3)} \leqslant \varepsilon_0$$

there is a uniquely determined energy solution

$$u \in \Big(\mathcal{C}\big([0,\infty), H^{s+1}\big(\mathbb{R}^3\big)\big) \cap \mathcal{C}^1\big([0,\infty), H^s\big(\mathbb{R}^3\big)\big)\Big)^3$$

to the Cauchy problem (3.1) *with* $\theta \in [0,1/2)$. *Moreover, the estimates for the solutions are the same as in Theorem 3.3.3.*

Proof. Firstly, we define the evolution space and its norm as in (3.24) and (3.25), respectively. Discussing the global (in time) existence of solutions with large regular data, we may use the fractional powers rules [92] instead of the fractional chain rule and the fractional Leibniz rule. More precisely, the estimates of $|u^{k-1}(\tau,\cdot)|^{p_k}$ in the $\dot{H}^s$ norm should be changed. By our assumption, which implies $\min\{p_1;p_2;p_3\}>s$, we have the following estimate:

$$\begin{aligned}\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)} &\lesssim \big\|u^{k-1}(\tau,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}\big\|u^{k-1}(\tau,\cdot)\big\|^{p_k-1}_{L^\infty(\mathbb{R}^3)}\\ &\lesssim \big\|u^{k-1}(\tau,\cdot)\big\|^{p_k}_{\dot{H}^s(\mathbb{R}^3)}+\big\|u^{k-1}(\tau,\cdot)\big\|^{p_k-1}_{\dot{H}^{s^*}(\mathbb{R}^3)}\big\|u^{k-1}(\tau,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)},\end{aligned}$$

where we applied Proposition B.2.7 with $0<2s^*<3<2s$.

Using the fractional Gagliardo-Nirenberg inequality again we can show

$$\begin{aligned}\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)} \lesssim& \big\|u^{k-1}(\tau,\cdot)\big\|^{\frac{p_k}{s+1}}_{L^2(\mathbb{R}^3)}\big\|u^{k-1}(\tau,\cdot)\big\|^{\frac{p_k s}{s+1}}_{\dot{H}^{s+1}(\mathbb{R}^3)}\\ &+\big\|u^{k-1}(\tau,\cdot)\big\|^{p_k-\frac{s^*}{s+1}p_k+\frac{s^*-s}{s+1}}_{L^2(\mathbb{R}^3)}\big\|u^{k-1}(\tau,\cdot)\big\|^{\frac{s^*}{s+1}p_k-\frac{s^*-s}{s+1}}_{\dot{H}^{s+1}(\mathbb{R}^3)}.\end{aligned}$$

So, we derive the estimate

$$\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)} \lesssim (1+\tau)^{-\rho_0(1,\theta)p_k-\left(\frac{s^*}{s+1}p_k-\frac{s^*-s}{s+1}\right)(\rho_{s+1}(1,\theta)-\rho_0(1,\theta))}\|u\|^{p_k}_{X(\tau)}.$$

Now we choose $s^*=3/2-\bar{\epsilon}$ with an arbitrary small constant $\bar{\epsilon}>0$. Therefore, in order to obtain (3.9), the exponents p_1,p_2,p_3 should fulfill

$$\max\{2;s\}<\min\{p_1;p_2;p_3\}. \tag{3.31}$$

To verify the Lipschitz condition (3.10), taking $\min\{p_1;p_2;p_3\}>1+s$, we have

$$\begin{aligned}&\big\||u^{k-1}(\tau,\cdot)|^{p_k}-|v^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)}\\ &\lesssim \big\|u^{k-1}(\tau,\cdot)-v^{k-1}(\tau,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}\int_0^1\big\|g\big(\nu u^{k-1}(\tau,\cdot)+(1-\nu)v^{k-1}(\tau,\cdot)\big)\big\|_{L^\infty(\mathbb{R}^3)}\mathrm{d}\nu\\ &\quad+\big\|u^{k-1}(\tau,\cdot)-v^{k-1}(\tau,\cdot)\big\|_{L^\infty(\mathbb{R}^3)}\int_0^1\big\|g\big(\nu u^{k-1}(\tau,\cdot)+(1-\nu)v^{k-1}(\tau,\cdot)\big)\big\|_{\dot{H}^s(\mathbb{R}^3)}\mathrm{d}\nu\\ &\lesssim \big\|u^{k-1}(\tau,\cdot)-v^{k-1}(\tau,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}\int_0^1\big\|\nu u^{k-1}(\tau,\cdot)+(1-\nu)v^{k-1}(\tau,\cdot)\big\|^{p_k-1}_{L^\infty(\mathbb{R}^3)}\mathrm{d}\nu\\ &\quad+\big\|u^{k-1}(\tau,\cdot)-v^{k-1}(\tau,\cdot)\big\|_{L^\infty(\mathbb{R}^3)}\int_0^1\big\|\nu u^{k-1}(\tau,\cdot)+(1-\nu)v^{k-1}(\tau,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}\\ &\qquad\qquad\times\big\|\nu u^{k-1}(\tau,\cdot)+(1-\nu)v^{k-1}(\tau,\cdot)\big\|^{p_k-2}_{L^\infty(\mathbb{R}^3)}\mathrm{d}\nu.\end{aligned}$$

After applying Proposition B.2.7 again, we can conclude (3.10) with the condition

$$1+s=\max\{2;1+s\}<\min\{p_1;p_2;p_3\}\quad\text{for}\quad s>\tfrac{3}{2}. \tag{3.32}$$

The proof is completed. □

Remark 3.3.6. *If one uses the embedding $H^s(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ for $s>3/2$, in other words,*

$$\|f\|_{L^\infty(\mathbb{R}^3)}\leqslant\|f\|_{L^2(\mathbb{R}^3)}+\|f\|_{\dot{H}^s(\mathbb{R}^3)}$$

to estimate the term $\big\|u^{k-1}(\tau,\cdot)\big\|_{L^\infty(\mathbb{R}^3)}$, then we obtain

$$\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{\dot{H}^s(\mathbb{R}^3)}\lesssim(1+\tau)^{-\left(p_k-\frac{s}{s+1}\right)\rho_0(1,\theta)-\frac{s}{s+1}\rho_{s+1}(1,\theta)}\|u\|^{p_k}_{X(\tau)}.$$

So, the global (in time) Sobolev solution exists uniquely with an additional restriction

$$2+\frac{3}{3-4\theta}<\min\{p_1;p_2;p_3\},$$

where $\theta\in[0,1/2)$.

3.4. Existence results for semilinear elastic waves with structural damping for $\theta \in [1/2,1]$

In the following we will prove results for global (in time) existence of small data Sobolev solutions to the weakly coupled system (3.1) with $\theta \in [1/2,1]$ and initial data belonging to $\mathcal{D}^s_{m,1}(\mathbb{R}^3)$ for some regularity parameters $m \in [1,2)$ and $s \geqslant 0$.

3.4.1. Data from classical energy space with suitable regularity

In this subsection we mainly study global (in time) existence of energy solutions with small data having an additional regularity L^m for $m \in [1,3/2)$. The main reason of the suitable choice of regularity $m \in [1,3/2)$ will be explained in Remark 3.4.1.

Firstly, we state our result for $m \in [1,6/5)$. As explained in Remark 3.3.2, if the exponents $p_{k_j} = p_c(m,\theta)$ in one of the Cases (ii) or (iii) for some $j = 1,2,3$ in the next theorem, then we can choose the parameters g_{k_j} describing the loss of decay as $g_{k_j} = \varepsilon_1$ with a sufficiently small constant $\varepsilon_1 > 0$.

Theorem 3.4.1. *Let us assume that the parameters $p_k \in [2/m,3]$ with $m \in [1,6/5)$ for $k = 1,2,3$, and satisfy one of the following conditions:*
(i) we assume

$$\min\{p_1;p_2;p_3\} > p_c(m,\theta); \tag{3.33}$$

(ii) we assume $\alpha_{k_1}(m,\theta) < 3/2$ when

$$1 < p_{k_1} < p_c(m,\theta) \quad \textit{and} \quad p_{k_2}, p_{k_3} > p_c(m,\theta); \tag{3.34}$$

(iii) we assume $\widetilde{\alpha}_{k_1}(m,\theta) < 3/2$ when

$$1 < p_{k_1}, p_{k_2} < p_c(m,\theta) \quad \textit{and} \quad p_{k_3} > p_c(m,\theta). \tag{3.35}$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for initial data $\big(u_0^k,u_1^k\big) \in \mathcal{D}^0_{m,1}(\mathbb{R}^3)$ with

$$\big\|\big(u_0^1,u_1^1\big)\big\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)} + \big\|\big(u_0^2,u_1^2\big)\big\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)} + \big\|\big(u_0^3,u_1^3\big)\big\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)} \leqslant \varepsilon_0$$

there is a uniquely determined energy solution

$$u \in \big(\mathcal{C}\big([0,\infty),H^1(\mathbb{R}^3)\big) \cap \mathcal{C}^1\big([0,\infty),L^2(\mathbb{R}^3)\big)\big)^3$$

to the Cauchy problem (3.1) *with $\theta \in [1/2,1]$. Moreover, the following estimates hold:*

$$\begin{aligned}
\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} &\lesssim (1+t)^{-\frac{6-5m}{4m\theta}+g_k} \sum_{k=1}^{3} \big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)},\\
\big\|\nabla u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} + \big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} &\lesssim (1+t)^{-\frac{6-3m}{4m\theta}+g_k} \sum_{k=1}^{3} \big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)},
\end{aligned}$$

where the parameters g_k are chosen in the following way:

1. *$g_k = 0$ for $k = 1,2,3$, when p_1,p_2,p_3 satisfy the condition* (3.33)*;*
2. *$g_{k_1} = \frac{3+2m\theta}{2m\theta} - \frac{3-m}{2m\theta}p_{k_1}$ and $g_{k_2} = g_{k_3} = 0$, when p_{k_1},p_{k_2},p_{k_3} satisfy the condition* (3.34)*;*
3. *$g_{k_1} = \frac{3+2m\theta}{2m\theta} - \frac{3-m}{2m\theta}p_{k_1}$, $g_{k_2} = \frac{3+2m\theta}{2m\theta} + \frac{1+2\theta}{2\theta}p_{k_2} - \frac{3-m}{2m\theta}p_{k_1}p_{k_2}$ and $g_{k_3} = 0$ when p_{k_1},p_{k_2},p_{k_3} satisfy the condition* (3.35).

Proof. We define for $T>0$ the spaces of solutions $X(T)$ by

$$X(T) := \big(\mathcal{C}\big([0,T],H^1(\mathbb{R}^3)\big)\cap\mathcal{C}^1\big([0,T],L^2(\mathbb{R}^3)\big)\big)^3 \tag{3.36}$$

with the corresponding norm

$$\begin{aligned}\|u\|_{X(T)} := \sup_{t\in[0,T]}\Big(&\sum_{k=1}^3(1+t)^{\frac{6-5m}{4m\theta}-g_k}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\sum_{k=1}^3(1+t)^{\frac{6-3m}{4m\theta}-g_k}\big\|\nabla u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&+\sum_{k=1}^3(1+t)^{\frac{6-3m}{4m\theta}-g_k}\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\Big).\end{aligned}$$

The classical Gagliardo-Nirenberg inequality implies

$$\begin{aligned}\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^m(\mathbb{R}^3)}&\lesssim(1+\tau)^{-\frac{(3-m)p_k-3}{2m\theta}+g_{k-1}p_k}\|u\|^{p_k}_{X(\tau)},\\\big\||u^{k-1}(\tau,\cdot)|^{p_k}\big\|_{L^2(\mathbb{R}^3)}&\lesssim(1+\tau)^{-\frac{2(3-m)p_k-3m}{4m\theta}+g_{k-1}p_k}\|u\|^{p_k}_{X(\tau)},\end{aligned}$$

where $m\in[1,6/5)$ and the parameters $\beta_{0,1}(mp_k)\in[0,1]$ and $\beta_{0,1}(2p_k)\in[0,1]$ in the classical Gagliardo-Nirenberg inequality lead to the condition $p_k\in[2/m,3]$ for all $k=1,2,3$.

To begin with, we apply the derived $\big(L^2\cap L^m\big)-L^2$ estimate on $[0,t]$ to estimate the solution as follows:

$$\begin{aligned}&(1+t)^{\frac{6-5m}{4m\theta}-g_k}\big\|u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&\lesssim(1+t)^{-g_k}\|u\|^{p_k}_{X(t)}\Big(\int_0^{t/2}(1+\tau)^{-\frac{(3-m)p_k-3}{2m\theta}+g_{k-1}p_k}\mathrm{d}\tau+(1+t)^{1-\frac{(3-m)p_k-3}{2m\theta}+g_{k-1}p_k}\Big),\end{aligned}$$

where we divided the interval $[0,t]$ into sub-intervals $[0,t/2]$ and $[t/2,t]$ and used

$$\int_{t/2}^t(1+t-\tau)^{-\frac{6-5m}{4m\theta}}\mathrm{d}\tau\lesssim(1+t)^{1-\frac{6-5m}{4m\theta}}$$

due to the fact that $6-5m<4m\theta$ for all $m\in[1,6/5)$ and $\theta\in[1/2,1]$.
Next, using the derived $\big(L^2\cap L^m\big)-L^2$ estimate on $[0,t/2]$ and L^2-L^2 estimate on $[t/2,t]$, we obtain the estimate for the first-order derivatives ($j+l=1$) as follows:

$$\begin{aligned}&(1+t)^{\frac{6-3m}{4m\theta}-g_k}\big\|\partial_t^j\nabla^l u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&\lesssim(1+t)^{-g_k}\|u\|^{p_k}_{X(t)}\Big(\int_0^{t/2}(1+\tau)^{-\frac{(3-m)p_k-3}{2m\theta}+g_{k-1}p_k}\mathrm{d}\tau+(1+t)^{1-\frac{(3-m)p_k-3}{2m\theta}+g_{k-1}p_k}\Big).\end{aligned}$$

Summarizing the above estimates gives

$$\begin{aligned}&(1+t)^{\frac{6-5m+2(j+l)m}{4m\theta}-g_k}\big\|\partial_t^j\nabla^l u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\&\lesssim(1+t)^{-g_k}\|u\|^{p_k}_{X(t)}\int_0^{t/2}(1+\tau)^{-\frac{(3-m)p_k-3}{2m\theta}+g_{k-1}p_k}\mathrm{d}\tau\\&\quad+(1+t)^{1-\frac{(3-m)p_k-3}{2m\theta}+g_{k-1}p_k-g_k}\|u\|^{p_k}_{X(t)}\end{aligned}\tag{3.37}$$

for all $j+l=0,1$ with $j,l\in\mathbb{N}_0$. Now, we distinguish between three cases to prove

$$(1+t)^{\frac{6-5m+2(j+l)m}{4m\theta}-g_k}\big\|\partial_t^j\nabla^l u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim\|u\|^{p_k}_{X(t)}. \tag{3.38}$$

Case 1: We assume the condition (3.33), that is, $\min\{p_1;p_2;p_3\}>p_c(m,\theta)$.

In this case, we have no loss of decay. So, we choose the parameters

$$g_1 = g_2 = g_3 = 0.$$

Therefore, (3.37) implies the following estimates for $j + l = 0, 1$:

$$\begin{aligned}&(1+t)^{\frac{6-5m+2(j+l)m}{4m\theta}} \big\|\partial_t^j \nabla^l u^k_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|u\|^{p_k}_{X(t)} \Big(\int_0^{t/2} (1+\tau)^{-\frac{(3-m)p_k-3}{2m\theta}} \mathrm{d}\tau + (1+t)^{1-\frac{(3-m)p_k-3}{2m\theta}} \Big),\end{aligned}$$

where $k = 1, 2, 3$. To prove (3.38), we need that the right-hand side in the last inequality is uniformly bounded in $t > 0$. But, this follows from the condition

$$\min\{p_1; p_2; p_3\} > p_c(m,\theta) = \frac{2m\theta + 3}{3-m} \quad \text{for} \quad \theta \in \big[\tfrac{1}{2}, 1\big].$$

Case 2: We assume the condition (3.34), that is, $1 < p_{k_1} < p_c(m,\theta)$ and $p_{k_2}, p_{k_3} > p_c(m,\theta)$.

Now, we allow a loss of decay in one component of the solution. We choose

$$g_{k_1} = \frac{3 + 2m\theta}{2m\theta} - \frac{3-m}{2m\theta} p_{k_1} \quad \text{and} \quad g_{k_2} = g_{k_3} = 0.$$

Obviously, by the assumption $1 < p_{k_1} < p_c(m,\theta)$ we may get $g_{k_1} > 0$. The condition $\alpha_{k_1}(m,\theta) < 3/2$ is equivalent to the following inequality:

$$3 + 2m\theta + m(1+2\theta)p_{k_2} + (m-3)p_{k_1}p_{k_2} < 0. \tag{3.39}$$

With the assumption $1 < p_{k_1} < p_c(m,\theta)$, the condition (3.39) is valid only when $p_{k_2} > p_c(m,\theta)$. Moreover, we have

$$(1+t)^{\frac{6-5m+2(j+l)m}{4m\theta} - g_{k_1}} \big\|\partial_t^j \nabla^l u^{k_1}_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{1-g_{k_1}-\frac{(3-m)p_{k_1}-3}{2m\theta}} \|u\|^{p_{k_1}}_{X(t)}, \tag{3.40}$$

where we used

$$\int_0^{t/2} (1+\tau)^{-\frac{(3-m)p_{k_1}-3}{2m\theta}} \mathrm{d}\tau \lesssim (1+t)^{1-\frac{(3-m)p_{k_1}-3}{2m\theta}}.$$

Hence, the above estimates lead to the desired estimate (3.38) when $k = k_1$.
When $k = k_2$, we obtain the following estimate:

$$\begin{aligned}&(1+t)^{\frac{6-5m+2(j+l)m}{4m\theta}} \big\|\partial_t^j \nabla^l u^{k_2}_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|u\|^{p_{k_2}}_{X(t)} \Big(\int_0^{t/2} (1+\tau)^{-\frac{(3-m)p_{k_2}-3}{2m\theta} + g_{k_1}p_{k_2}} \mathrm{d}\tau + (1+t)^{1-\frac{(3-m)p_{k_2}-3}{2m\theta} + g_{k_1}p_{k_2}} \Big).\end{aligned}$$

Applying the condition (3.39) it follows

$$-\frac{(3-m)p_{k_2} - 3}{2m\theta} + g_{k_1}p_{k_2} = \frac{3 + m(1+2\theta)p_{k_2} + (m-3)p_{k_1}p_{k_2}}{2m\theta} < -1.$$

So it immediately leads to the estimate (3.38) when $k = k_2$.
The case $k = k_3$ can be treated by using the same arguments as we did in studying *Case 1*. More precisely, we have

$$\begin{aligned}&(1+t)^{\frac{6-5m+2(j+l)m}{4m\theta}} \big\|\partial_t^j \nabla^l u^{k_3}_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|u\|^{p_{k_3}}_{X(t)} \Big(\int_0^{t/2} (1+\tau)^{-\frac{(3-m)p_{k_3}-3}{2m\theta}} \mathrm{d}\tau + (1+t)^{1-\frac{(3-m)p_{k_3}-3}{2m\theta}} \Big).\end{aligned}$$

Taking account of $p_{k_3} > p_c(m,\theta)$, the estimate (3.38) is valid for $k = k_3$.

Case 3: We assume the condition (3.35), that is, $1 < p_{k_1}, p_{k_2} < p_c(m,\theta)$ and $p_{k_3} > p_c(m,\theta)$. Here we take the parameters describing the loss of decay as follows:

$$g_{k_1} = \frac{3+2m\theta}{2m\theta} - \frac{3-m}{2m\theta}p_{k_1}, \quad g_{k_2} = \frac{3+2m\theta}{2m\theta} + \frac{1+2\theta}{2\theta}p_{k_2} - \frac{3-m}{2m\theta}p_{k_1}p_{k_2} \quad \text{and} \quad g_{k_3} = 0.$$

With the help of $1 < p_{k_1}, p_{k_2} < p_c(m,\theta)$, we conclude that $g_{k_1} > 0$ as well as $g_{k_2} > 0$. Then, the condition $\widetilde{\alpha}_{k_1}(m,\theta) < 3/2$ can be rewritten as

$$3 + 2m\theta + m(1+2\theta)p_{k_3}(1+p_{k_2}) + (m-3)p_{k_1}p_{k_2}p_{k_3} < 0. \tag{3.41}$$

We know that the inequality (3.41) is valid only if $p_{k_3} > p_c(m,\theta)$. Following the same approach for treating *Case 2* we immediately obtain the desired estimate (3.38) in this case.

Lastly, no matter in which case, we may derive the Lipschitz condition by using Hölder's inequality and the Gagliardo-Nirenberg inequality. Then, we may prove

$$(1+t)^{\frac{6-5m+2(j+l)m}{4m\theta}-g_k}\left\|\partial_t^j\nabla^l\left(u^k_{\text{non}} - v^k_{\text{non}}\right)(t,\cdot)\right\|_{L^2(\mathbb{R}^3)} \lesssim \|u-v\|_{X(t)}\left(\|u\|^{p_k-1}_{X(t)} + \|v\|^{p_k-1}_{X(t)}\right)$$

for $j+l = 0,1$ with $j,l \in \mathbb{N}_0$ and $k = 1,2,3$ for *Cases 1-3*. Therefore, the proof is complete. □

Next, in the case when $m \in [6/5, 3/2)$, the estimates for the solutions to the Cauchy problem (2.1) are different to those in the case $m \in [1, 6/5)$. For this reason we feel differences in estimating the norms $\left\||u^{k-1}(\tau,\cdot)|^{p_k}\right\|_{L^m(\mathbb{R}^3)}$ and $\left\||u^{k-1}(\tau,\cdot)|^{p_k}\right\|_{L^2(\mathbb{R}^3)}$.

We now state our result for $m \in [6/5, 3/2)$. If some of the exponents $p_{k_j} = p_{\text{bal}}(m,0,\theta)$ for some $j = 1,2,3$, we can choose the parameters g_{k_j} describing the loss of decay as $g_{k_j} = \varepsilon_1$ with a sufficiently small constant $\varepsilon_1 > 0$.

Theorem 3.4.2. *Let us assume $p_k \in [2/m, 3]$ with $m \in [6/5, 3/2)$ for $k = 1,2,3$, and satisfy one of the following conditions:*
(i) we assume

$$\min\{p_1; p_2; p_3\} > p_{\text{bal}}(m,0,\theta); \tag{3.42}$$

(ii) we assume $\alpha_{k_1,\text{bal}}(m,0,\theta) < 3/2$ when

$$1 < p_{k_1} < p_{\text{bal}}(m,0,\theta) \quad \textit{and} \quad p_{k_2}, p_{k_3} > p_{\text{bal}}(m,0,\theta); \tag{3.43}$$

(iii) we assume $\widetilde{\alpha}_{k_1,\text{bal}}(m,0,\theta) < 3/2$ when

$$1 < p_{k_1}, p_{k_2} < p_{\text{bal}}(m,0,\theta) \quad \textit{and} \quad p_{k_3} > p_{\text{bal}}(m,0,\theta). \tag{3.44}$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for initial data $\left(u_0^k, u_1^k\right) \in \mathcal{D}^0_{m,1}(\mathbb{R}^3)$ with

$$\left\|\left(u_0^1, u_1^1\right)\right\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)} + \left\|\left(u_0^2, u_1^2\right)\right\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)} + \left\|\left(u_0^3, u_1^3\right)\right\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)} \leqslant \varepsilon_0$$

there is a uniquely determined energy solution

$$u \in \left(\mathcal{C}\left([0,\infty), H^1(\mathbb{R}^3)\right) \cap \mathcal{C}^1\left([0,\infty), L^2(\mathbb{R}^3)\right)\right)^3$$

to the Cauchy problem (3.1) *with $\theta \in [1/2, 1]$. Moreover, the following estimates hold:*

$$\left\|u^k(t,\cdot)\right\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{1-\frac{6-3m}{4m\theta}+g_k}\sum_{k=1}^3\left\|\left(u_0^k, u_1^k\right)\right\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)},$$

$$\left\|\nabla u^k(t,\cdot)\right\|_{L^2(\mathbb{R}^3)} + \left\|u_t^k(t,\cdot)\right\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{6-3m}{4m\theta}+g_k}\sum_{k=1}^3\left\|\left(u_0^k, u_1^k\right)\right\|_{\mathcal{D}^0_{m,1}(\mathbb{R}^3)},$$

where the parameters g_k are chosen in the following way:

1. $g_k = 0$ *for* $k = 1, 2, 3$, *when* p_1, p_2, p_3 *satisfy the condition* (3.42)*;*

2. $g_{k_1} = \frac{m+3}{m} - \big(\frac{1}{2} + \frac{6-3m}{4m\theta}\big)p_{k_1}$ *and* $g_{k_2} = g_{k_3} = 0$, *when* $p_{k_1}, p_{k_2}, p_{k_3}$ *satisfy the condition* (3.43)*;*

3. $g_{k_1} = \frac{m+3}{m} - \big(\frac{1}{2} + \frac{6-3m}{4m\theta}\big)p_{k_1}$, $g_{k_2} = \frac{m+3}{m} - \big(\frac{6-3m}{4m\theta} - \frac{1}{2} - \frac{3}{m}\big)p_{k_2} - \big(\frac{1}{2} + \frac{6-3m}{4m\theta}\big)p_{k_1}p_{k_2}$ *and* $g_{k_3} = 0$ *when* $p_{k_1}, p_{k_2}, p_{k_3}$ *satisfy the condition* (3.44).

Proof. Here we only redefine the norm of the evolution space (3.36) as follows:

$$\begin{aligned}\|u\|_{X(T)} := \sup_{t\in[0,T]}\Big(&\sum_{k=1}^{3}(1+t)^{-1+\frac{6-3m}{4m\theta}-g_k}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\\ &+\sum_{k=1}^{3}(1+t)^{\frac{6-3m}{4m\theta}-g_k}\big(\big\|\nabla u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\big)\Big).\end{aligned}$$

After following the proof of Theorem 3.4.1 we may conclude the desired statements. □

Remark 3.4.1. *If we would consider the Cauchy problem model* (3.1) *with* $\theta \in [1/2, 1]$ *and* $m \in [3/2, 2)$, *we should guarantee* $\max\{p_1; p_2; p_3\} \leqslant 3$ *from parameter restrictions generating by the application of the Gagliardo-Nirenberg inequality. But, we should give at least for one exponent* $k = 1, 2, 3$ *another restriction*

$$p_k > p_{\text{bal}}(m, 0, \theta) = 2 + \frac{6(m-2+2\theta)}{2m\theta - 3m + 6} \geqslant 3 \quad \text{if} \quad m \in \big[\tfrac{3}{2}, 2\big) \quad \text{and} \quad \theta \in \big[\tfrac{1}{2}, 1\big].$$

In conclusion, the set of admissible triplets of exponents (p_1, p_2, p_3) *is empty.*

3.4.2. Data from energy space with suitable higher regularity

Theorem 3.4.3. *Let us choose* $m \in [1, 6/5)$ *and assume that the exponents satisfy*

$$\begin{aligned}&1+\lceil s\rceil < \min\{p_1;p_2;p_3\} \leqslant \max\{p_1;p_2;p_3\} \leqslant 1+\frac{2}{1-2s} && \text{if } s\in\big(0,\tfrac{1}{2}\big),\\ &1+\lceil s\rceil < \min\{p_1;p_2;p_3\} \leqslant \max\{p_1;p_2;p_3\} < \infty && \text{if } s\in\big[\tfrac{1}{2},\infty\big).\end{aligned}$$

The exponents p_1, p_2, p_3 *satisfy one of the conditions* (3.33) *to* (3.35)*. Then, there exists a constant* $\varepsilon_0 > 0$ *such that for initial data* $\big(u_0^k, u_1^k\big) \in \mathcal{D}^s_{m,1}\big(\mathbb{R}^3\big)$ *with*

$$\big\|\big(u_0^1, u_1^1\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)} + \big\|\big(u_0^2, u_1^2\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)} + \big\|\big(u_0^3, u_1^3\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)} \leqslant \varepsilon_0$$

there is a uniquely determined energy solution

$$u \in \big(\mathcal{C}\big([0,\infty), H^{s+1}\big(\mathbb{R}^3\big)\big) \cap \mathcal{C}^1\big([0,\infty), H^s\big(\mathbb{R}^3\big)\big)\big)^3$$

to the Cauchy problem (3.1) *with* $\theta \in [1/2, 1]$*. Moreover, the following estimates hold:*

$$\begin{aligned}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} &\lesssim (1+t)^{-\frac{6-5m}{4m\theta}+g_k}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)},\\ \big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)} &\lesssim (1+t)^{-\frac{6-3m}{4m\theta}+g_k}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)},\\ \big\||D|u^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}+\big\|u_t^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)} &\lesssim (1+t)^{-\frac{6-3m+2sm}{4m\theta}+g_k}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)},\end{aligned}$$

where the parameters g_k *are the same as in Theorem 3.4.1.*

Proof. We define the norm for the evolution space (3.24) as follows:

$$\|u\|_{X(T)} := \sup_{t\in[0,T]}\Big(\sum_{k=1}^{3}(1+t)^{\frac{6-5m}{4m\theta}-g_k}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\sum_{k=1}^{3}(1+t)^{\frac{6-3m}{4m\theta}-g_k}\big\|u^k_t(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}$$
$$+\sum_{k=1}^{3}(1+t)^{\frac{6-3m+2sm}{4m\theta}-g_k}\big(\big\||D|u^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}+\big\|u^k_t(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}\big)\Big).$$

Then, we immediately follow the proof of Theorem 3.3.3 to complete this proof. □

The choice of initial data from higher-order energy spaces allows us to weaken the upper bounds for the exponents p_1, p_2, p_3 for which we can prove the global (in time) existence of small data energy solutions by choosing the parameter of additional regularity in the interval $m\in[3/2,2)$. To be more precise, the following statements hold:

$$\Big[\frac{2}{m},1+\frac{2}{1-2s}\Big]\cap\big(p_{\text{bal}}(m,0,\theta),\infty\big)\neq\emptyset \quad\text{for}\quad s\in\big(0,\tfrac{1}{2}\big),$$
$$\Big[\frac{2}{m},\infty\Big)\cap\big(p_{\text{bal}}(m,0,\theta),\infty\big)\neq\emptyset \quad\text{for}\quad s\in\big[\tfrac{1}{2},\infty\big),$$

when $\theta\in[1/2,1]$ and $m\in[3/2,2)$. Thus, we can get a more flexible (with respect to s) admissible range of exponents p_1, p_2, p_3.

Theorem 3.4.4. *Let us choose $m\in[6/5,2)$ and assume that the exponents satisfy*

$$1+\lceil s\rceil<\min\{p_1;p_2;p_3\}\leqslant\max\{p_1;p_2;p_3\}\leqslant 1+\frac{2}{1-2s}\qquad \textit{if}\quad s\in\big(0,\tfrac{1}{2}\big),$$
$$1+\lceil s\rceil<\min\{p_1;p_2;p_3\}\leqslant\max\{p_1;p_2;p_3\}<\infty\qquad \textit{if}\quad s\in\big[\tfrac{1}{2},\infty\big).$$

The exponents p_1, p_2, p_3 satisfy one of the conditions (3.42) *to* (3.44)*. Then, there exists a constant $\varepsilon_0>0$ such that for initial data $\big(u_0^k,u_1^k\big)\in\mathcal{D}^s_{m,1}\big(\mathbb{R}^3\big)$ with*

$$\big\|\big(u_0^1,u_1^1\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)}+\big\|\big(u_0^2,u_1^2\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)}+\big\|\big(u_0^3,u_1^3\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)}\leqslant\varepsilon_0$$

there is a uniquely determined energy solution

$$u\in\Big(\mathcal{C}\big([0,\infty),H^{s+1}\big(\mathbb{R}^3\big)\big)\cap\mathcal{C}^1\big([0,\infty),H^{s}\big(\mathbb{R}^3\big)\big)\Big)^3$$

to the Cauchy problem (3.1) *with $\theta\in[1/2,1]$. Moreover, the following estimates hold:*

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim(1+t)^{1-\frac{6-3m}{4m\theta}+g_k}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)},$$
$$\big\|u^k_t(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim(1+t)^{-\frac{6-3m}{4m\theta}+g_k}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)},$$
$$\big\||D|u^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}+\big\|u^k_t(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}\lesssim(1+t)^{-\frac{6-3m+2sm}{4m\theta}+g_k}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)},$$

where the parameters g_k are the same as in Theorem 3.4.1.

Proof. Following the proof of Theorem 3.3.3, this theorem can be proved immediately by redefining the norm for the evolution space (3.24) as follows:

$$\|u\|_{X(T)} := \sup_{t\in[0,T]}\Big(\sum_{k=1}^{3}(1+t)^{-1+\frac{6-3m}{4m\theta}-g_k}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}+\sum_{k=1}^{3}(1+t)^{\frac{6-3m}{4m\theta}-g_k}\big\|u^k_t(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}$$
$$+\sum_{k=1}^{3}(1+t)^{\frac{6-3m+2sm}{4m\theta}-g_k}\big(\big\||D|u^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}+\big\|u^k_t(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}\big)\Big),$$

where g_k are the same as in Theorem 3.4.1. □

Finally, we are interested in the case of large regular data belonging to L^∞, too. For this reason, we choose the regularity parameter s from the interval $(3/2,\infty)$. Let us restrict ourselves to the case $m\in[1,6/5)$.

Theorem 3.4.5. *Let us choose $m\in[1,6/5)$, $s>3/2$ and assume that the exponents satisfy*

$$\max\{1+s;p_c(m,\theta)\}<\min\{p_1;p_2;p_3\},$$

and one of the conditions (3.33) *to* (3.35)*. Then, there exists a positive constant $\varepsilon_0>0$ such that for initial data $\big(u_0^k,u_1^k\big)\in\mathcal{D}^s_{m,1}\big(\mathbb{R}^3\big)$ with*

$$\big\|\big(u_0^1,u_1^1\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)}+\big\|\big(u_0^2,u_1^2\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)}+\big\|\big(u_0^3,u_1^3\big)\big\|_{\mathcal{D}^s_{m,1}(\mathbb{R}^3)}\leqslant\varepsilon_0$$

there is a uniquely determined energy solution

$$u\in\Big(\mathcal{C}\big([0,\infty),H^{s+1}\big(\mathbb{R}^3\big)\big)\cap\mathcal{C}^1\big([0,\infty),H^s\big(\mathbb{R}^3\big)\big)\Big)^3$$

to the Cauchy problem (3.1) *with $\theta\in[1/2,1]$. Moreover, the estimates for the solutions are the same as in Theorem 3.4.3 after choosing $g_k=0$ for all $k=1,2,3$.*

Proof. One can complete the proof by following the same steps of the proof of Theorem 3.3.5. □

Remark 3.4.2. *Providing that one uses the embedding $H^s\big(\mathbb{R}^3\big)\hookrightarrow L^\infty\big(\mathbb{R}^3\big)$ for $s>3/2$, we should give the following condition:*

$$1+s<\min\{p_1;p_2;p_3\}\quad\textit{and}\quad\frac{4m\theta+6-3m}{6-5m}\leqslant\min\{p_1;p_2;p_3\},$$

to prove the global (in time) existence of small data Sobolev solution.

3.5. Existence and restriction of parameters

The purpose of this section is to clarify the possibility to choose the parameters $q_1,q_2,r_1,\dots,r_6$ in the proof of Theorem 3.3.3. In other words, we show that the condition (3.45) below is not only a sufficient condition but also a necessary condition for a suitable choice of these parameters.

First of all, in Theorem 3.3.3 the restrictions for the parameters q_1 and q_2 are as follows:

$$\frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{q_1}\Big)\in[0,1],\quad\frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{q_2}+\frac{s}{3}\Big)\in\Big[\frac{s}{s+1},1\Big]\quad\text{and}\quad\frac{p_k-1}{q_1}+\frac{1}{q_2}=\frac{1}{2}$$

with $q_1,q_2\neq\infty$. In other words, we have

$$\begin{aligned}\frac{(1-2s)(p_k-1)+1}{6}&\leqslant\frac{p_k-1}{q_1}+\frac{1}{q_2}\leqslant\frac{p_k}{2}\quad\text{if } 0<s<\tfrac{1}{2},\\ \frac{1}{6}&\leqslant\frac{p_k-1}{q_1}+\frac{1}{q_2}\leqslant\frac{p_k}{2}\quad\text{if } \tfrac{1}{2}\leqslant s.\end{aligned}$$

Hence, the assumption

$$p_k\leqslant1+\frac{2}{1-2s}\quad\text{if } 0<s<\tfrac{1}{2}\tag{3.45}$$

for p_k for all $k=1,2,3$, leads to

$$\frac{1}{2}\in\Big[\frac{(1-2s)(p_k-1)+1}{6},\frac{p_k}{2}\Big]\quad\text{and}\quad\frac{1}{2}\in\Big[\frac{1}{6},\frac{p_k}{2}\Big].$$

All in all, (3.45) is a necessary condition to choose suitable parameters q_1 and q_2.

Next, we want to show that the condition (3.45) is a sufficient condition for choosing q_1, q_2. From the relationship

$$\frac{1}{q_2}=\frac{1}{2}-\frac{p_k-1}{q_1} \quad \text{and} \quad \frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{q_1}\Big)\in[0,1],$$

we may conclude

$$\begin{aligned} &\frac{1}{q_2}\in\Big[1-\frac{p_k}{2},\frac{1}{2}-\frac{(1-2s)(p_k-1)}{6}\Big] && \text{if } 0<s<\tfrac{1}{2},\\ &\frac{1}{q_2}\in\Big[1-\frac{p_k}{2},\frac{1}{2}\Big) && \text{if } \tfrac{1}{2}\leqslant s. \end{aligned} \tag{3.46}$$

We should point out that the interval for $\frac{1}{q_2}$ is not empty because of $p_k>1$. Taking account of $\frac{3}{s+1}\big(\frac{1}{2}-\frac{1}{q_2}+\frac{s}{3}\big)\in\big[\frac{s}{s+1},1\big]$ and (3.46) together with assumption (3.45), we observe

$$\begin{aligned} &\Big[\frac{1}{6},\frac{1}{2}\Big]\cap\Big[1-\frac{p_k}{2},\frac{1}{2}-\frac{(1-2s)(p_k-1)}{6}\Big]\neq\emptyset && \text{if } 0<s<\tfrac{1}{2},\\ &\Big(0,\frac{1}{2}\Big]\cap\Big[1-\frac{p_k}{2},\frac{1}{2}\Big)\neq\emptyset && \text{if } \tfrac{1}{2}\leqslant s. \end{aligned}$$

In conclusion, there exist suitable parameters q_1, q_2 in the proof of Theorem 3.3.3.

Moreover, the restrictions on r_1, r_2 are

$$\frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{r_1}+\frac{s}{3}\Big)\in\Big[\frac{s}{s+1},1\Big],\quad \frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{r_2(p_k-1)}\Big)\in[0,1],\quad\text{and}\quad \frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{2}.$$

By the same arguments as above, the condition (3.45) is a sufficient and necessary condition for the choice of suitable parameters r_1 and r_2.

Lastly, the restrictions on $r_3,\dots,r_6$ are

$$\begin{aligned} &\frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{r_3}\Big)\in[0,1],\quad \frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{r_5}\Big)\in[0,1],\quad \frac{3}{s+1}\Big(\frac{1}{2}-\frac{1}{r_6}+\frac{s}{3}\Big)\in\Big[\frac{s}{s+1},1\Big],\\ &\frac{1}{r_3}+\frac{1}{r_4}=\frac{1}{2}\quad\text{and}\quad \frac{1}{r_4}=\frac{p_k-2}{r_5}+\frac{1}{r_6}. \end{aligned}$$

As in the paper [81] we also prove the optimality of the condition (3.45) for p_k.

One choice for the parameters q_1, q_2 and $r_1,\dots,r_6$ is the following:

$$\begin{aligned} &q_1=3(p_k-1),\ q_2=6,\ r_1=6,\ r_2=3,\ r_3=3(p_k-1),\\ &r_4=\frac{6(p_k-1)}{3(p_k-1)-2},\ r_5=3(p_k-1),\ r_6=6. \end{aligned}$$

This choice implies the following condition for $k=1,2,3$:

$$1+\frac{2}{3}\leqslant p_k\leqslant 1+\frac{2}{1-2s}\quad\text{if}\quad 0<s<\tfrac{1}{2}.$$

3.6. Concluding remarks

Remark 3.6.1. *If $\theta\in[0,1/2)$ in* (3.1) *we proved in this chapter the global (in time) existence of energy solutions with small initial data belonging to $\mathcal{D}^s_{1,1}\big(\mathbb{R}^3\big)$ or $\mathcal{D}^s_{3/2,1}\big(\mathbb{R}^3\big)$. One may also consider the global (in time) existence of solutions with $\big(u_0^k,u_1^k\big)\in\mathcal{D}^s_{m,1}\big(\mathbb{R}^3\big)$ for $m\in[1,2)$, $s\geqslant 0$ by using the energy estimates to the linear model* (2.1) *(c.f. with Theorems 2.4.5).*

Remark 3.6.2. *In Section 3.4 we proved some results for the global (in time) existence of small data solutions to some semilinear models with exponents p_1,p_2,p_3 satisfying some conditions. Up to now, we did not prove any optimality of the exponent for the global (in time) existence of small data solutions.*

But we expect that the following exponents and parameters are critical to the Cauchy problem (3.1) *with a structural damping* $(-\Delta)^{1/2}u_t$*:*

$$\begin{aligned}
&p_c\big(1,\tfrac{1}{2}\big)=2,\\
&\alpha_{\max}\big(1,\tfrac{1}{2}\big)=\max\big\{\alpha_1\big(1,\tfrac{1}{2}\big);\alpha_2\big(1,\tfrac{1}{2}\big)\big\}=\frac{3}{2},\\
&\widetilde{\alpha}_{\max}\big(1,\tfrac{1}{2}\big)=\max\big\{\widetilde{\alpha}_1\big(1,\tfrac{1}{2}\big);\widetilde{\alpha}_2\big(1,\tfrac{1}{2}\big);\widetilde{\alpha}_3\big(1,\tfrac{1}{2}\big)\big\}=\frac{3}{2}.
\end{aligned}$$

The main reason is that the exponent $p_c\big(1,\frac{1}{2}\big)$ *corresponds to the critical exponent to the semilinear structurally damped wave equation (which was studied in [24])*

$$\begin{cases} u_{tt}-\Delta u+(-\Delta)^{1/2}u_t=|u|^p, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbb{R}^3. \end{cases}$$

The recent paper [19] proved a global (in time) existence result to the weakly coupled system

$$\begin{cases} u_{tt}-\Delta u+(-\Delta)^{1/2}u_t=|v|^{p_1}, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ v_{tt}-\Delta v+(-\Delta)^{1/2}v_t=|u|^{p_2}, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ (u,u_t,v,v_t)(0,x)=(u_0,u_1,v_0,v_1)(x), & x\in\mathbb{R}^3, \end{cases}$$

when the exponents p_1,p_2 *satisfy the condition* $\alpha_{\max}\big(1,\frac{1}{2}\big)>3/2$ *and a blow-up result when the condition* $\alpha_{\max}\big(1,\frac{1}{2}\big)<3/2$ *holds. Thus, we conjecture that the parameter* $\alpha_{\max}\big(1,\frac{1}{2}\big)=3/2$ *is critical if only one exponent is below or equal to the exponent* $p_c\big(1,\frac{1}{2}\big)$*. In addition, as mentioned in Section 3.1, the parameter* $\widetilde{\alpha}_{\max}\big(1,\frac{1}{2}\big)$ *is naturally generalized from the parameter* $\alpha_{\max}\big(1,\frac{1}{2}\big)$*. Therefore, we also conjecture that the parameter* $\widetilde{\alpha}_{\max}\big(1,\frac{1}{2}\big)=3/2$ *is critical when two exponents are below or equal to the exponent* $p_c\big(1,\frac{1}{2}\big)$*.*

4. Elastic waves with Kelvin-Voigt damping in 2D and 3D

4.1. Introduction

In this chapter we mainly study the following Cauchy problem for linear elastic waves with *Kelvin-Voigt damping* (c.f. [71, 65]):

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + \mathbb{E}u_t = 0, & x \in \mathbb{R}^2,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^2, \end{cases} \tag{4.1}$$

with Lamé operator

$$\mathbb{E} := -a^2\Delta - \big(b^2 - a^2\big)\nabla \operatorname{div} \quad \text{carrying} \quad b > a > 0,$$

where $u = \big(u^1, u^2\big)^{\mathrm{T}}$, and the weakly coupled system of semilinear elastic waves with Kelvin-Voigt damping in 2D, that is, the Cauchy problem (4.1) with nonlinear terms

$$f(u) := \big(|u^2|^{p_1}, |u^1|^{p_2}\big)^{\mathrm{T}}$$

on the right-hand sides (one can see (4.37) later). Here the exponents satisfy $p_1, p_2 > 1$. Furthermore, we also study the elastic waves with Kelvin-Voigt damping in 3D.

In the case when $a^2 = b^2$ in the Cauchy problem (4.1), the Kelvin-Voigt damped elastic waves will be transfered to the viscoelastic damped wave equation. For viscoelastic damped wave equation, one can derive estimates of solutions and asymptotic profiles of solutions by using explicit representations of solutions in the Fourier space (for example, one may see [24, 25, 40]). However, due to the Lamé operator $\mathbb{E}$ appearing in (4.1), we should use some decomposition method to treat elastic waves with Kelvin-Voigt damping. For the three dimensional case, we may use the Helmholtz decomposition to decompose the solution to a potential part and a solenoidal part. Nevertheless, this decomposition does not hold any more in the two dimensional case. Concerning energy estimates for the Cauchy problem (4.1), the recent paper [109] derived almost sharp energy estimates by using energy methods in the Fourier space. But, sharp energy estimates and diffusion phenomena for two dimensional elastic waves with Kelvin-Voigt damping are not clear. Strongly motivated by the pioneering paper [108], we employ energy methods in the Fourier space to derive energy estimates for the linear Cauchy problem (4.1). Then, by applying the spectral theory associated with asymptotic expansions of eigenvalues and their corresponding eigenprojections, the sharpness of the derived energy estimates have been shown. Basing on the representation of solutions, we investigate diffusion phenomena for the Cauchy problem (4.48). Then, by employing $L^2 - L^2$ estimates with additional L^m regularity and Banach's fixed-point theorem, we prove global (in time) existence of small data solutions to the weakly coupled system of semilinear elastic waves with Kelvin-Voigt damping in 2D and 3D, respectively.

The rest of this chapter is organized as follows. We first investigate some qualitative properties of solutions to the two dimensional linear Cauchy problem (4.1) from Section 4.2 to Section 4.4. Specifically, we prepare pointwise estimates of solutions in the Fourier space by applying Lemma 2.2 in [108] in Section 4.2. Then, we obtain energy estimates of solutions to the Kelvin-Voigt damped elastic waves (4.1). In Subsection 4.3.1 we derive asymptotic expansions of eigenvalues and their corresponding eigenprojections to show the sharpness of derived pointwise estimates. After constructing asymptotic representations of solutions in Subsection 4.3.2, we derive diffusion phenomena in Section 4.4, where initial data is taken from L^m spaces with $m \in [1, 2]$. Then, in Section 4.5 we prove global (in time) existence of small data solutions to the weakly coupled system of semilinear elastic waves with Kelvin-Voigt damping. Additionally, the three dimensional elastic waves with Kelvin-Voigt damping is treated in Section 4.6. Finally, in Section 4.7 some concluding remarks complete this chapter.

4.2. Estimates of solutions to the linear problem in 2D

In this section we will derive estimates of solutions to the linear elastic waves with Kelvin-Voigt damping in two dimensions.

Indeed, the system in (4.1) can be rewritten in the following form:

$$u_{tt} - a^2\Delta(u+u_t) - \left(b^2-a^2\right)\begin{pmatrix} \partial_{x_1}^2 & \partial_{x_1x_2} \\ \partial_{x_2x_1} & \partial_{x_2}^2 \end{pmatrix}(u+u_t) = 0. \tag{4.2}$$

Applying the partial Fourier transformation with respect to spatial variables to (4.2), i.e., $\hat{u}(t,\xi) = \mathcal{F}_{x\to\xi}(u(t,x))$ gives

$$\hat{u}_{tt} + |\xi|^2 A(\eta)\hat{u}_t + |\xi|^2 A(\eta)\hat{u} = 0, \tag{4.3}$$

where $\eta = \xi/|\xi| \in \mathbb{S}^1$ and

$$A(\eta) = \begin{pmatrix} a^2 + \left(b^2-a^2\right)\eta_1^2 & \left(b^2-a^2\right)\eta_1\eta_2 \\ \left(b^2-a^2\right)\eta_1\eta_2 & a^2 + \left(b^2-a^2\right)\eta_2^2 \end{pmatrix}.$$

Because of our assumption $b > a > 0$, the matrix $A(\eta)$ is positive definite. The eigenvalues of $A(\eta)$ are b^2 and a^2. We introduce the matrices

$$M(\eta) := \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_2 & -\eta_1 \end{pmatrix} \quad \text{and} \quad A_{\text{diag}}(|\xi|) := |\xi|^2 \text{diag}\left(b^2, a^2\right).$$

By using the change of variables

$$v(t,\xi) := M^{-1}(\eta)\hat{u}(t,\xi),$$

we define

$$W(t,\xi) := \begin{pmatrix} v_t(t,\xi) + iA_{\text{diag}}^{1/2}(|\xi|)v(t,\xi) \\ v_t(t,\xi) - iA_{\text{diag}}^{1/2}(|\xi|)v(t,\xi) \end{pmatrix}.$$

According to the above matrices we can directly compute

$$\begin{aligned} W_t &= \begin{pmatrix} -|\xi|^2 M^{-1}(\eta)A(\eta)M(\eta)(v_t+v) + iA_{\text{diag}}^{1/2}(|\xi|)v_t \\ -|\xi|^2 M^{-1}(\eta)A(\eta)M(\eta)(v_t+v) - iA_{\text{diag}}^{1/2}(|\xi|)v_t \end{pmatrix} \\ &= \begin{pmatrix} -A_{\text{diag}}(|\xi|)(v_t+v) + iA_{\text{diag}}^{1/2}(|\xi|)v_t \\ -A_{\text{diag}}(|\xi|)(v_t+v) - iA_{\text{diag}}^{1/2}(|\xi|)v_t \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}A_{\text{diag}}(|\xi|) + iA_{\text{diag}}^{1/2}(|\xi|) & -\frac{1}{2}A_{\text{diag}}(|\xi|) \\ -\frac{1}{2}A_{\text{diag}}(|\xi|) & -\frac{1}{2}A_{\text{diag}}(|\xi|) - iA_{\text{diag}}^{1/2}(|\xi|) \end{pmatrix} W. \end{aligned}$$

Therefore, we derive the evolution system

$$\begin{cases} W_t + \frac{1}{2}|\xi|^2 B_0 W - i|\xi| B_1 W = 0, & \xi \in \mathbb{R}^2,\ t \in \mathbb{R}_+, \\ W(0,\xi) = W_0(\xi), & \xi \in \mathbb{R}^2, \end{cases} \tag{4.4}$$

where the coefficient matrices B_0 and B_1 are given by

$$B_0 = \begin{pmatrix} b^2 & 0 & b^2 & 0 \\ 0 & a^2 & 0 & a^2 \\ b^2 & 0 & b^2 & 0 \\ 0 & a^2 & 0 & a^2 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}.$$

The solution to (4.4) is given by

$$W(t,\xi) = \mathrm{e}^{t\hat{\Phi}(|\xi|)} W_0(\xi),$$

where

$$\hat{\Phi}(|\xi|) = -\frac{1}{2}|\xi|^2 B_0 + i|\xi| B_1 \\ = \begin{pmatrix} -\frac{b^2}{2}|\xi|^2 + ib|\xi| & 0 & -\frac{b^2}{2}|\xi|^2 & 0 \\ 0 & -\frac{a^2}{2}|\xi|^2 + ia|\xi| & 0 & -\frac{a^2}{2}|\xi|^2 \\ -\frac{b^2}{2}|\xi|^2 & 0 & -\frac{b^2}{2}|\xi|^2 - ib|\xi| & 0 \\ 0 & -\frac{a^2}{2}|\xi|^2 & 0 & -\frac{a^2}{2}|\xi|^2 - ia|\xi| \end{pmatrix}. \tag{4.5}$$

Moreover, the eigenvalue problem corresponding to (4.4) is

$$\lambda\phi + \Big(\frac{1}{2}|\xi|^2 B_0 - i|\xi| B_1\Big)\phi = 0, \tag{4.6}$$

where $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{C}^4$. The eigenvalue $\lambda = \lambda(|\xi|)$ of the problem (4.4) is the value of λ satisfying (4.6) for $\phi \neq 0$.

We now derive energy estimates basing on pointwise estimates of the partial Fourier transform of solutions in the Fourier space. In the pioneering paper [108] the authors derived pointwise estimates for a general class of symmetric hyperbolic-parabolic systems by using the energy method in the Fourier space. Thus, we may apply Lemma 2.2 in [108] to complete the following lemma.

Lemma 4.2.1. *The solution $W = W(t,\xi)$ to the Cauchy problem* (4.4) *satisfies the following pointwise estimates for any $\xi \in \mathbb{R}^2$ and $t \geqslant 0$:*

$$|W(t,\xi)| \lesssim \mathrm{e}^{-c\rho(|\xi|)t} |W_0(\xi)|, \tag{4.7}$$

where

$$\rho(|\xi|) := \frac{|\xi|^2}{1+|\xi|^2}$$

and c is a positive constant.

Remark 4.2.1. *From the asymptotic behavior of eigenvalues $\lambda_j(|\xi|)$ for $j = 1,\dots,4$, which will be shown later in* (4.13) *and* (4.17)*, the dissipative structure of the system* (4.4) *can be characterized by the property*

$$\mathrm{Re}\lambda_j(|\xi|) \leqslant -c\rho(|\xi|).$$

Moreover, according to the asymptotic expansions of eigenvalues for $|\xi| \to 0$ and $|\xi| \to \infty$, our pointwise estimates of the partial Fourier transform of solutions stated in Lemma 4.2.1 are sharp.

Proof. Let us define a real and antisymmetric matrix $\widetilde{K}$ by

$$\widetilde{K} := \begin{pmatrix} 0 & 0 & -\frac{b}{2} & 0 \\ 0 & 0 & 0 & -\frac{a}{2} \\ \frac{b}{2} & 0 & 0 & 0 \\ 0 & \frac{a}{2} & 0 & 0 \end{pmatrix}.$$

The coefficient matrices in (4.4), i.e., $\frac{1}{2}B_0$ and $-B_1$ are real symmetric. Moreover, the matrix $\frac{1}{2}B_0$ is positive semi-definite. According to Lemma 2.2 in [108], due to the fact that $-\widetilde{K}B_1 + \frac{1}{2}B_0$ is positive definite, we can derive

$$|W(t,\xi)| \lesssim \mathrm{e}^{-\frac{c|\xi|^2}{1+|\xi|^2}t} |W_0(\xi)|,$$

for all $t \geqslant 0$ and $\xi \in \mathbb{R}^2$, with a positive constant c. $\square$

Next, we concentrate on the estimates of the classical energy and higher-order energy of solutions with initial data taking from $H^s(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)$. Because of the proof is quite standard, we only sketch it.

Theorem 4.2.1. *Let us consider the Cauchy problem* (4.1) *and initial data satisfies*

$$\big(|D|u_0^k, u_1^k\big) \in \big(H^s(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)\big) \times \big(H^s(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)\big)$$

for $k = 1, 2$*, where* $s \geqslant 0$ *and* $m \in [1, 2]$*. Then, we have the following estimates for the energies of higher-order:*

$$\begin{aligned}
&\big\||D|^{s+1}u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^2)} + \big\||D|^s u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^2)} \\
&\lesssim (1+t)^{-\frac{2-m+ms}{2m}} \sum_{k=1}^{2} \big\|\big(|D|u_0^k, u_1^k\big)\big\|_{(H^s(\mathbb{R}^2)\cap L^m(\mathbb{R}^2))\times(H^s(\mathbb{R}^2)\cap L^m(\mathbb{R}^2))}.
\end{aligned}$$

Proof. We can derive the following estimates by applying the Parseval-Plancherel theorem and the pointwise estimates of the partial Fourier transform of solutions in the Fourier space in Lemma 4.2.1:

$$\begin{aligned}
\big\||D|^s \mathcal{F}^{-1}_{\xi\to x}(W)(t,\cdot)\big\|_{(L^2(\mathbb{R}^2))^4} &= \big\||\xi|^s W(t,\xi)\big\|_{(L^2(\mathbb{R}^2))^4} \\
&\lesssim \big\||\xi|^s \mathrm{e}^{-c\rho(|\xi|)t} W_0(\xi)\big\|_{(L^2(\mathbb{R}^2))^4} \\
&\lesssim \big\|\chi_{\mathrm{int}}(\xi)|\xi|^s \mathrm{e}^{-c|\xi|^2 t} W_0(\xi)\big\|_{(L^2(\mathbb{R}^2))^4} \\
&\quad + \big\|(\chi_{\mathrm{mid}}(\xi) + \chi_{\mathrm{ext}}(\xi))|\xi|^s \mathrm{e}^{-ct} W_0(\xi)\big\|_{(L^2(\mathbb{R}^2))^4}.
\end{aligned} \tag{4.8}$$

For small frequencies, we apply Hölder's inequality and the Hausdorff-Young inequality to get

$$\big\|\chi_{\mathrm{int}}(\xi)|\xi|^s \mathrm{e}^{-c|\xi|^2 t} W_0(\xi)\big\|_{(L^2(\mathbb{R}^2))^4} \lesssim (1+t)^{-\frac{2-m+ms}{2m}} \big\|\mathcal{F}^{-1}(W_0)\big\|_{(L^m(\mathbb{R}^2))^4}.$$

For middle and large frequencies, we immediately obtain

$$\big\|(\chi_{\mathrm{mid}}(\xi) + \chi_{\mathrm{ext}}(\xi))|\xi|^s \mathrm{e}^{-ct} W_0(\xi)\big\|_{(L^2(\mathbb{R}^2))^4} \lesssim \mathrm{e}^{-ct} \big\|\mathcal{F}^{-1}(W_0)\big\|_{(H^s(\mathbb{R}^2))^4}.$$

Together them completes the proof. □

Remark 4.2.2. *To derive the estimate of the solution itself to the Cauchy problem* (4.1) *with*

$$\big(|D|u_0^k, u_1^k\big) \in \big(L^2(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)\big) \times \big(L^2(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)\big),$$

we need to estimate

$$\big\|\chi_{\mathrm{int}}(\xi)|\xi|^{-1} W(t,\xi)\big\|_{(L^2(\mathbb{R}^2))^4}.$$

Then, using Hölder's inequality, we have to derive an estimate for the following term:

$$\big\|\chi_{\mathrm{int}}(\xi)|\xi|^{-1} \mathrm{e}^{-c|\xi|^2 t}\big\|_{L^{\frac{2m}{2-m}}(\mathbb{R}^2)}.$$

Nevertheless, due to a strong influence of the singularity for $|\xi| \to +0$*, i.e., non integrability, the following inequality does not hold for all* $t \geqslant 0$ *and* $m \in [1, 2]$*:*

$$\big\|\chi_{\mathrm{int}}(\xi)|\xi|^{-1} \mathrm{e}^{-c|\xi|^2 t}\big\|_{L^{\frac{2m}{2-m}}(\mathbb{R}^2)} < \infty.$$

Remark 4.2.3. *It is not reasonable for us to compare the estimates in Theorem 4.2.1 with those in [109] due to different assumptions of the data spaces. In the recent paper [109], the authors proved estimates for the classical energy to* (4.1) *with initial data is taken from* $\big(H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)\big) \times \big(L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)\big)$ *by using the energy method in the Fourier space and the Haraux-Komornik inequality.*

4.3. Asymptotic expansions and asymptotic representations

With the aim of determining whether the pointwise estimates in the Fourier space in Lemma 4.2.1 are sharp or not, we now investigate the asymptotic expansion of eigenvalue of (4.4) for $|\xi| \to 0$ and $|\xi| \to \infty$. Moreover, to derive diffusion phenomena, we need to get the asymptotic expressions of the propagator $\mathrm{e}^{t\hat{\Phi}(|\xi|)}$ (see the definition in (4.5)). This method is strongly motivated from the paper [39].

4.3.1. Asymptotic expansions

We denote by $\lambda_j = \lambda_j(|\xi|)$, $j = 1, \dots, 4$, the eigenvalues of matrix $\hat{\Phi}(|\xi|)$. Thus, these eigenvalues are the solutions to the following characteristic equation:

$$\begin{aligned} F(\lambda) &= \det\big(\lambda I_{4\times 4} - \hat{\Phi}(|\xi|)\big) \\ &= \begin{vmatrix} \lambda + \frac{b^2}{2}|\xi|^2 - ib|\xi| & 0 & \frac{b^2}{2}|\xi|^2 & 0 \\ 0 & \lambda + \frac{a^2}{2}|\xi|^2 - ia|\xi| & 0 & \frac{a^2}{2}|\xi|^2 \\ \frac{b^2}{2}|\xi|^2 & 0 & \lambda + \frac{b^2}{2}|\xi|^2 + ib|\xi| & 0 \\ 0 & \frac{a^2}{2}|\xi|^2 & 0 & \lambda + \frac{a^2}{2}|\xi|^2 + ia|\xi| \end{vmatrix} \\ &= \lambda^4 + \big(a^2+b^2\big)|\xi|^2\lambda^3 + \big(\big(a^2+b^2\big)|\xi|^2 + a^2b^2|\xi|^4\big)\lambda^2 + 2a^2b^2|\xi|^4\lambda + a^2b^2|\xi|^4, \end{aligned} \tag{4.9}$$

where $|\xi|$ is regarded as a parameter. We notice that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda) = 4\lambda^3 + 3\big(a^2+b^2\big)|\xi|^2\lambda^2 + 2\big(\big(a^2+b^2\big)|\xi|^2 + a^2b^2|\xi|^4\big)\lambda + 2a^2b^2|\xi|^4.$$

Comparing the polynomials $F(\lambda)$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda)$, we observe that there are non-trivial common divisors at most for values of frequencies $|\xi|$ in a zero measure set. Thus, we have only simple roots of $F(\lambda) = 0$ outside of this zero measure set.

Let $P_j(|\xi|)$ be the corresponding eigenprojections, which can be expressed as

$$P_j(|\xi|) = \prod_{k\neq j} \frac{\hat{\Phi}(|\xi|) - \lambda_k(|\xi|)I_{4\times 4}}{\lambda_j(|\xi|) - \lambda_k(|\xi|)}. \tag{4.10}$$

In the next step we distinguish the asymptotic expansions of eigenvalues and their corresponding eigenprojections between two cases: $|\xi| \to 0$ and $|\xi| \to \infty$. These expansions essentially determine the asymptotic behavior of solutions.

Asymptotic expansions for $|\xi| \to 0$

We deduce that the eigenvalues $\lambda_j(|\xi|)$ and their corresponding eigenprojections $P_j(|\xi|)$ have the following asymptotic expansions for $|\xi| \to 0$, respectively:

$$\lambda_j(|\xi|) = \lambda_j^{(0)} + \lambda_j^{(1)}|\xi| + \lambda_j^{(2)}|\xi|^2 + \cdots, \tag{4.11}$$

$$P_j(|\xi|) = P_j^{(0)} + P_j^{(1)}|\xi| + P_j^{(2)}|\xi|^2 + \cdots, \tag{4.12}$$

where $\lambda_j^{(k)} \in \mathbb{C}$, $P_j^{(k)} \in \mathbb{C}^{4\times 4}$ for all $k \in \mathbb{N}$.

Then, we substitute $\lambda = \lambda_j(|\xi|)$ chosen in (4.11) into the characteristic equation (4.9) and calculate the coefficients $\lambda_j^{(k)}$. After lengthy but straightforward calculations, the value of pairwise distinct coefficients are given by

$$\begin{cases} \lambda_1^{(0)} = \lambda_2^{(0)} = \lambda_3^{(0)} = \lambda_4^{(0)} = 0, \\ \lambda_1^{(1)} = ib, \quad \lambda_2^{(1)} = -ib, \quad \lambda_3^{(1)} = ia, \quad \lambda_4^{(1)} = -ia, \\ \lambda_1^{(2)} = \lambda_2^{(2)} = -\frac{b^2}{2}, \quad \lambda_3^{(2)} = \lambda_4^{(2)} = -\frac{a^2}{2}. \end{cases}$$

Consequently, the eigenvalues have the following asymptotic behaviors for $|\xi| \to 0$:

$$\begin{aligned}
\lambda_1(|\xi|) &= ib|\xi| - \frac{b^2}{2}|\xi|^2 + O\big(|\xi|^3\big), \quad & \lambda_3(|\xi|) &= ia|\xi| - \frac{a^2}{2}|\xi|^2 + O\big(|\xi|^3\big),\\
\lambda_2(|\xi|) &= -ib|\xi| - \frac{b^2}{2}|\xi|^2 + O\big(|\xi|^3\big), & \lambda_4(|\xi|) &= -ia|\xi| - \frac{a^2}{2}|\xi|^2 + O\big(|\xi|^3\big).
\end{aligned} \tag{4.13}$$

By using the pairwise distinct eigenvalues given in (4.13) and the matrix $\hat{\Phi}(|\xi|)$ given in (4.5), we employ (4.10) to calculate $P_j^{(0)}$.
For the case when $j = 1$ in (4.12), we have

$$\prod_{k\neq 1}(\lambda_1(|\xi|) - \lambda_k(|\xi|)) = -2ib\big(b^2 - a^2\big)|\xi|^3 + O\big(|\xi|^4\big),$$

$$\prod_{k\neq 1}\big(\hat{\Phi}(|\xi|) - \lambda_k(|\xi|)I_{4\times 4}\big) = \begin{pmatrix} b_{11}^{(1)} & 0 & b_{13}^{(1)} & 0\\ 0 & b_{22}^{(1)} & 0 & b_{24}^{(1)}\\ b_{31}^{(1)} & 0 & b_{33}^{(1)} & 0\\ 0 & b_{42}^{(1)} & 0 & b_{44}^{(1)} \end{pmatrix},$$

where

$$\begin{aligned}
b_{11}^{(1)} &= -2ib\big(b^2 - a^2\big)|\xi|^3 + O\big(|\xi|^4\big), & b_{13}^{(1)} &= \frac{b^2}{2}\big(b^2 - a^2\big)|\xi|^4 + O\big(|\xi|^5\big),\\
b_{22}^{(1)} &= \frac{ia^4}{4}(a+b)|\xi|^5 + O\big(|\xi|^6\big), & b_{24}^{(1)} &= -\frac{a^6}{8}|\xi|^6,\\
b_{31}^{(1)} &= \frac{b^2}{2}\big(b^2 - a^2\big)|\xi|^4 + O\big(|\xi|^5\big), & b_{33}^{(1)} &= \frac{b^4}{4}\big(a^2 - b^2\big)|\xi|^6,\\
b_{42}^{(1)} &= -\frac{a^6}{8}|\xi|^6, & b_{44}^{(1)} &= \frac{ia^4}{4}(b-a)|\xi|^5 + O\big(|\xi|^6\big).
\end{aligned}$$

For the case when $j = 2$ in (4.12), we have

$$\prod_{k\neq 2}(\lambda_2(|\xi|) - \lambda_k(|\xi|)) = 2ib\big(b^2 - a^2\big)|\xi|^3 + O\big(|\xi|^4\big),$$

$$\prod_{k\neq 2}\big(\hat{\Phi}(|\xi|) - \lambda_k(|\xi|)I_{4\times 4}\big) = \begin{pmatrix} b_{11}^{(2)} & 0 & b_{13}^{(2)} & 0\\ 0 & b_{22}^{(2)} & 0 & b_{24}^{(2)}\\ b_{31}^{(2)} & 0 & b_{33}^{(2)} & 0\\ 0 & b_{42}^{(2)} & 0 & b_{44}^{(2)} \end{pmatrix},$$

where

$$\begin{aligned}
b_{11}^{(2)} &= \frac{b^4}{4}\big(a^2 - b^2\big)|\xi|^6, & b_{13}^{(2)} &= \frac{b^2}{2}\big(b^2 - a^2\big)|\xi|^4 + O\big(|\xi|^5\big),\\
b_{22}^{(2)} &= \frac{ia^4}{4}(a-b)|\xi|^5 + O\big(|\xi|^6\big), & b_{24}^{(2)} &= -\frac{a^6}{8}|\xi|^6,\\
b_{31}^{(2)} &= \frac{b^2}{2}\big(b^2 - a^2\big)|\xi|^4 + O\big(|\xi|^5\big), & b_{33}^{(2)} &= 2ib\big(b^2 - a^2\big)|\xi|^3 + O\big(|\xi|^4\big),\\
b_{42}^{(2)} &= -\frac{a^6}{8}|\xi|^6, & b_{44}^{(2)} &= -\frac{ia^4}{4}(a+b)|\xi|^5 + O\big(|\xi|^6\big).
\end{aligned}$$

For the case when $j = 3$ in (4.12), we have

$$\prod_{k\neq 3}(\lambda_3(|\xi|) - \lambda_k(|\xi|)) = -2ia\big(a^2 - b^2\big)|\xi|^3 + O\big(|\xi|^4\big),$$

$$\prod_{k\neq 3}\big(\hat{\Phi}(|\xi|) - \lambda_k(|\xi|)I_{4\times 4}\big) = \begin{pmatrix} b_{11}^{(3)} & 0 & b_{13}^{(3)} & 0 \\ 0 & b_{22}^{(3)} & 0 & b_{24}^{(3)} \\ b_{31}^{(3)} & 0 & b_{33}^{(3)} & 0 \\ 0 & b_{42}^{(3)} & 0 & b_{44}^{(3)} \end{pmatrix},$$

where

$$\begin{aligned} b_{11}^{(3)} &= \frac{ib^4}{4}(a+b)|\xi|^5 + O\big(|\xi|^6\big), & b_{13}^{(3)} &= -\frac{b^6}{8}|\xi|^6, \\ b_{22}^{(3)} &= -2ia\big(a^2 - b^2\big)|\xi|^3 + O\big(|\xi|^4\big), & b_{24}^{(3)} &= \frac{a^2}{2}\big(a^2 - b^2\big)|\xi|^4 + O\big(|\xi|^5\big), \\ b_{31}^{(3)} &= -\frac{b^4}{8}|\xi|^6, & b_{33}^{(3)} &= \frac{ib^4}{4}(a-b)|\xi|^5 + O\big(|\xi|^6\big), \\ b_{42}^{(3)} &= \frac{a^2}{2}\big(a^2 - b^2\big)|\xi|^4 + O\big(|\xi|^5\big), & b_{44}^{(3)} &= \frac{a^4}{4}\big(b^2 - a^2\big)|\xi|^6. \end{aligned}$$

For the case when $j = 4$ in (4.12), we have

$$\prod_{k\neq 4}(\lambda_4(|\xi|) - \lambda_k(|\xi|)) = 2ia\big(a^2 - b^2\big)|\xi|^3 + O\big(|\xi|^4\big),$$

$$\prod_{k\neq 4}\big(\hat{\Phi}(|\xi|) - \lambda_k(|\xi|)I_{4\times 4}\big) = \begin{pmatrix} b_{11}^{(4)} & 0 & b_{13}^{(4)} & 0 \\ 0 & b_{22}^{(4)} & 0 & b_{24}^{(4)} \\ b_{31}^{(4)} & 0 & b_{33}^{(4)} & 0 \\ 0 & b_{42}^{(4)} & 0 & b_{44}^{(4)} \end{pmatrix},$$

where

$$\begin{aligned} b_{11}^{(4)} &= \frac{ib^4}{4}(b-a)|\xi|^5 + O\big(|\xi|^6\big), & b_{13}^{(4)} &= -\frac{b^6}{8}|\xi|^6, \\ b_{22}^{(4)} &= \frac{a^4}{4}\big(b^2 - a^2\big)|\xi|^6, & b_{24}^{(4)} &= \frac{a^2}{2}\big(a^2 - b^2\big)|\xi|^4 + O\big(|\xi|^5\big), \\ b_{31}^{(4)} &= -\frac{b^4}{8}|\xi|^6, & b_{33}^{(4)} &= -\frac{ib^4}{4}(a+b)|\xi|^5 + O\big(|\xi|^6\big), \\ b_{42}^{(4)} &= \frac{a^2}{2}\big(a^2 - b^2\big)|\xi|^4 + O\big(|\xi|^5\big), & b_{44}^{(4)} &= 2ia\big(a^2 - b^2\big)|\xi|^3 + O\big(|\xi|^4\big). \end{aligned}$$

In conclusion, we obtain

$$\begin{aligned} P_1^{(0)} &= \mathrm{diag}(1,0,0,0), & P_3^{(0)} &= \mathrm{diag}(0,1,0,0), \\ P_2^{(0)} &= \mathrm{diag}(0,0,1,0), & P_4^{(0)} &= \mathrm{diag}(0,0,0,1). \end{aligned} \tag{4.14}$$

Thus, we get

$$P_j(|\xi|) - P_j^{(0)} = O(|\xi|)$$

for $|\xi| \to 0$.

Asymptotic expansions for $|\xi| \to \infty$

Similarly, the eigenvalues $\lambda_j(|\xi|)$ and their corresponding eigenprojections $P_j(|\xi|)$ have the following asymptotic expansions for $|\xi| \to \infty$, respectively:

$$\lambda_j(|\xi|) = \lambda_j^{(0)}|\xi|^2 + \lambda_j^{(1)}|\xi| + \lambda_j^{(2)} + \lambda_j^{(3)}|\xi|^{-1} + \lambda_j^{(4)}|\xi|^{-2} + \cdots, \tag{4.15}$$

$$P_j(|\xi|) = P_j^{(0)}|\xi|^2 + P_j^{(1)}|\xi| + P_j^{(2)} + P_j^{(3)}|\xi|^{-1} + P_j^{(4)}|\xi|^{-2} + \cdots, \tag{4.16}$$

where $\lambda_j^{(k)} \in \mathbb{C}$ and $P_j^{(k)} \in \mathbb{C}^{4\times 4}$ for all $k \in \mathbb{N}$.

We plug $\lambda = \lambda_j(|\xi|)$ chosen in (4.15) into the characteristic equation (4.9) to obtain the pairwise distinct value of coefficients $\lambda_j^{(k)}$

$$\begin{cases} \lambda_1^{(0)} = \lambda_2^{(0)} = 0, \quad \lambda_3^{(0)} = -b^2, \quad \lambda_4^{(0)} = -a^2, \\ \lambda_1^{(1)} = \lambda_2^{(1)} = \lambda_3^{(1)} = \lambda_4^{(1)} = 0, \\ \lambda_1^{(2)} = \lambda_2^{(2)} = -1, \quad \lambda_3^{(2)} = \lambda_4^{(2)} = 1, \\ \lambda_1^{(3)} = \lambda_2^{(3)} = \lambda_3^{(3)} = \lambda_4^{(3)} = 0, \\ \lambda_1^{(4)} = -\frac{1}{b^2}, \quad \lambda_2^{(4)} = -\frac{1}{a^2}, \quad \lambda_3^{(4)} = \frac{1}{b^2}, \quad \lambda_4^{(4)} = \frac{1}{a^2}. \end{cases}$$

Consequently, we investigate the following asymptotic behaviors of pairwise distinct eigenvalues $\lambda_j(|\xi|), j = 1, \ldots, 4$ for $|\xi| \to \infty$:

$$\begin{aligned} \lambda_1(|\xi|) &= -1 - \frac{1}{b^2}|\xi|^{-2} + O\big(|\xi|^{-3}\big), \quad \lambda_3(|\xi|) = -b^2|\xi|^2 + 1 + \frac{1}{b^2}|\xi|^{-2} + O\big(|\xi|^{-3}\big), \\ \lambda_2(|\xi|) &= -1 - \frac{1}{a^2}|\xi|^{-2} + O\big(|\xi|^{-3}\big), \quad \lambda_4(|\xi|) = -a^2|\xi|^2 + 1 + \frac{1}{a^2}|\xi|^{-2} + O\big(|\xi|^{-3}\big). \end{aligned} \tag{4.17}$$

By straightforward computations, we may derive represents of eigenprojections.
For the case when $j = 1$ in (4.16), we have

$$\prod_{k\neq 1}(\lambda_1(|\xi|) - \lambda_k(|\xi|)) = \big(b^2 - a^2\big)|\xi|^2 + O(|\xi|),$$

$$\prod_{k\neq 1}\big(\hat{\Phi}(|\xi|) - \lambda_k(|\xi|)I_{4\times 4}\big) = \begin{pmatrix} a_{11}^{(1)} & 0 & a_{13}^{(1)} & 0 \\ 0 & a_{22}^{(1)} & 0 & a_{24}^{(1)} \\ a_{31}^{(1)} & 0 & a_{33}^{(1)} & 0 \\ 0 & a_{42}^{(1)} & 0 & a_{44}^{(1)} \end{pmatrix},$$

where

$$\begin{aligned} a_{11}^{(1)} &= \frac{1}{2}\big(b^2 - a^2\big)|\xi|^2 + O(|\xi|), & a_{13}^{(1)} &= -\frac{1}{2}\big(b^2 - a^2\big)|\xi|^2 + O(1), \\ a_{22}^{(1)} &= \frac{1}{a^2}\big(a^2 - 2b^2\big) + O\big(|\xi|^{-1}\big), & a_{24}^{(1)} &= 1 + \frac{1}{2a^2|\xi|^2}, \\ a_{31}^{(1)} &= -\frac{1}{2}\big(b^2 - a^2\big)|\xi|^2 + O(1), & a_{33}^{(1)} &= \frac{1}{2}\big(b^2 - a^2\big)|\xi|^2 + O(|\xi|), \\ a_{42}^{(1)} &= 1 + \frac{1}{2a^2|\xi|^2}, & a_{44}^{(1)} &= \frac{1}{a^2}\big(a^2 - 2b^2\big) + O\big(|\xi|^{-1}\big). \end{aligned}$$

For the case when $j = 2$ in (4.16), we have

$$\prod_{k\neq 2}(\lambda_2(|\xi|) - \lambda_k(|\xi|)) = \big(a^2 - b^2\big)|\xi|^2 + O(|\xi|),$$

$$\prod_{k\neq 2}\big(\hat{\Phi}(|\xi|) - \lambda_k(|\xi|)I_{4\times 4}\big) = \begin{pmatrix} a_{11}^{(2)} & 0 & a_{13}^{(2)} & 0 \\ 0 & a_{22}^{(2)} & 0 & a_{24}^{(2)} \\ a_{31}^{(2)} & 0 & a_{33}^{(2)} & 0 \\ 0 & a_{42}^{(2)} & 0 & a_{44}^{(2)} \end{pmatrix},$$

where

$$\begin{aligned}
a_{11}^{(2)} &= \frac{1}{b^2}\big(b^2-2a^2\big)+O\big(|\xi|^{-1}\big), & a_{13}^{(2)} &= 1+\frac{1}{2b^2|\xi|^2},\\
a_{22}^{(2)} &= \frac{1}{2}\big(a^2-b^2\big)|\xi|^2+O(|\xi|), & a_{24}^{(2)} &= -\frac{1}{2}\big(a^2-b^2\big)|\xi|^2+O(1),\\
a_{31}^{(2)} &= 1+\frac{1}{2b^2|\xi|^2}, & a_{33}^{(2)} &= \frac{1}{b^2}\big(b^2-2a^2\big)+O\big(|\xi|^{-1}\big),\\
a_{42}^{(2)} &= -\frac{1}{2}\big(a^2-b^2\big)|\xi|^2+O(1), & a_{44}^{(2)} &= \frac{1}{2}\big(a^2-b^2\big)|\xi|^2+O(|\xi|).
\end{aligned}$$

For the case when $j=3$ in (4.16), we have

$$\prod_{k\neq 3}(\lambda_3(|\xi|)-\lambda_k(|\xi|)) = b^4\big(a^2-b^2\big)|\xi|^6+O\big(|\xi|^4\big),$$

$$\prod_{k\neq 3}\big(\hat{\Phi}(|\xi|)-\lambda_k(|\xi|)I_{4\times 4}\big) = \begin{pmatrix} a_{11}^{(3)} & 0 & a_{13}^{(3)} & 0\\ 0 & a_{22}^{(3)} & 0 & a_{24}^{(3)}\\ a_{31}^{(3)} & 0 & a_{33}^{(3)} & 0\\ 0 & a_{42}^{(3)} & 0 & a_{44}^{(3)} \end{pmatrix},$$

where

$$\begin{aligned}
a_{11}^{(3)} &= \frac{b^4}{2}\big(a^2-b^2\big)|\xi|^6+O\big(|\xi|^5\big), & a_{13}^{(3)} &= \frac{b^4}{2}\big(a^2-b^2\big)|\xi|^6+O\big(|\xi|^4\big),\\
a_{22}^{(3)} &= 1+O\big(|\xi|^{-1}\big), & a_{24}^{(3)} &= 1+\frac{1}{2a^2|\xi|^2},\\
a_{31}^{(3)} &= \frac{b^4}{2}\big(a^2-b^2\big)|\xi|^6+O\big(|\xi|^4\big)\big), & a_{33}^{(3)} &= \frac{b^4}{2}\big(a^2-b^2\big)|\xi|^6+O\big(|\xi|^5\big),\\
a_{42}^{(3)} &= 1+\frac{1}{2a^2|\xi|^2}, & a_{44}^{(3)} &= 1+O\big(|\xi|^{-1}\big).
\end{aligned}$$

For the case when $j=4$ in (4.16), we have

$$\prod_{k\neq 4}(\lambda_4(|\xi|)-\lambda_k(|\xi|)) = a^4\big(b^2-a^2\big)|\xi|^6+O\big(|\xi|^4\big),$$

$$\prod_{k\neq 4}\big(\hat{\Phi}(|\xi|)-\lambda_k(|\xi|)I_{4\times 4}\big) = \begin{pmatrix} a_{11}^{(4)} & 0 & a_{13}^{(4)} & 0\\ 0 & a_{22}^{(4)} & 0 & a_{24}^{(4)}\\ a_{31}^{(4)} & 0 & a_{33}^{(4)} & 0\\ 0 & a_{42}^{(4)} & 0 & a_{44}^{(4)} \end{pmatrix},$$

where

$$\begin{aligned}
a_{11}^{(4)} &= 1+O\big(|\xi|^{-1}\big), & a_{13}^{(4)} &= 1+\frac{1}{2b^2|\xi|^2},\\
a_{22}^{(4)} &= \frac{a^4}{2}\big(b^2-a^2\big)|\xi|^6+O\big(|\xi|^5\big), & a_{24}^{(4)} &= \frac{a^4}{2}\big(b^2-a^2\big)|\xi|^6+O\big(|\xi|^4\big),\\
a_{31}^{(4)} &= 1+\frac{1}{2b^2|\xi|^2}, & a_{33}^{(4)} &= 1+O\big(|\xi|^{-1}\big),\\
a_{42}^{(4)} &= \frac{a^4}{2}\big(b^2-a^2\big)|\xi|^6+O\big(|\xi|^4\big), & a_{44}^{(4)} &= \frac{a^4}{2}\big(b^2-a^2\big)|\xi|^6+O\big(|\xi|^5\big).
\end{aligned}$$

In conclusion, we obtain

$$
\begin{aligned}
&P_j^{(0)} = P_j^{(1)} = \mathbf{0}_{4\times 4} \quad \text{for} \quad j = 1,2,3,4,\\
&P_1^{(2)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_2^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1 \end{pmatrix},\\
&P_3^{(2)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_4^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 \end{pmatrix}.
\end{aligned}
\tag{4.18}
$$

It leads to

$$
P_j(|\xi|) - P_j^{(0)}|\xi|^2 - P_j^{(1)}|\xi| - P_j^{(2)} = O\big(|\xi|^{-1}\big)
$$

for $|\xi| \to \infty$.

4.3.2. Asymptotic representations

In the last subsection we have calculated the asymptotic expansions of eigenvalues $\lambda_j(|\xi|)$ and their corresponding eigenprojections $P_j(|\xi|)$ for $|\xi| \to 0$ and $|\xi| \to \infty$. In order to give the representation of solution $W(t,\xi) = \mathrm{e}^{t\hat{\Phi}(|\xi|)}W_0(\xi)$, motivated by [39], in this section we give asymptotic expressions of the propagator $\mathrm{e}^{t\hat{\Phi}(|\xi|)}$ for $|\xi| \to 0$ and $|\xi| \to \infty$.

The propagator $\mathrm{e}^{t\hat{\Phi}(|\xi|)}$ has the following spectral decomposition:

$$
\mathrm{e}^{t\hat{\Phi}(|\xi|)} = \sum_{j=1}^{4} \mathrm{e}^{\lambda_j(|\xi|)t} P_j(|\xi|), \tag{4.19}
$$

where $\lambda_j(|\xi|)$ are the eigenvalues of $\hat{\Phi}(|\xi|)$ and $P_j(|\xi|)$ are their corresponding eigenprojections. Now, we distinguish between two cases: $|\xi| \to 0$ and $|\xi| \to \infty$, respectively, to discuss the asymptotic expressions of (4.19).

Asymptotic expansion of $\mathrm{e}^{t\hat{\Phi}(|\xi|)}$ for $|\xi| \to 0$

We first define the matrix $\hat{S}_0(t,\xi)$ by

$$
\hat{S}_0(t,\xi) := \sum_{j=1}^{4} \mathrm{e}^{\lambda_j^0(|\xi|)t} P_j^{(0)}, \tag{4.20}
$$

where

$$
\begin{aligned}
\lambda_1^0(|\xi|) &= ib|\xi| - \frac{b^2}{2}|\xi|^2, \qquad \lambda_3^0(|\xi|) = ia|\xi| - \frac{a^2}{2}|\xi|^2,\\
\lambda_2^0(|\xi|) &= -ib|\xi| - \frac{b^2}{2}|\xi|^2, \quad \lambda_4^0(|\xi|) = -ia|\xi| - \frac{a^2}{2}|\xi|^2,
\end{aligned}
\tag{4.21}
$$

and $P_j^{(0)}$ are given in (4.14). We then rewrite the propagator for $|\xi| \to 0$ by

$$
\mathrm{e}^{t\hat{\Phi}(|\xi|)} = \hat{S}_0(t,\xi) + \hat{R}_0(t,\xi). \tag{4.22}
$$

Let us derive a pointwise estimate for the remainder $\hat{R}_0(t,\xi)$ for $|\xi| \to 0$.

Lemma 4.3.1. *We have the following estimate of remainder:*

$$\big|\hat{R}_0(t,\xi)\big| \lesssim |\xi|\mathrm{e}^{-c|\xi|^2t}, \tag{4.23}$$

where $0<|\xi|\leqslant\varepsilon$ *and* c *is a positive constant.*

Proof. From (4.19), (4.20) and (4.22), we can rewrite the remainder by

$$\begin{aligned}\hat{R}_0(t,\xi) &= \sum_{j=1}^4 \mathrm{e}^{\lambda_j(|\xi|)t}P_j(|\xi|) - \sum_{j=1}^4 \mathrm{e}^{\lambda_j^0(|\xi|)t}P_j^{(0)}\\ &= \sum_{j=1}^4 \mathrm{e}^{\lambda_j(|\xi|)t}\big(P_j(|\xi|)-P_j^{(0)}\big) + \sum_{j=1}^4 \mathrm{e}^{\lambda_j^0(|\xi|)t}\big(\mathrm{e}^{\lambda_j(|\xi|)t-\lambda_j^0(|\xi|)t}-1\big)P_j^{(0)}\\ &=: \hat{R}_{0,1}(t,\xi)+\hat{R}_{0,2}(t,\xi).\end{aligned}$$

Due to the facts that $P_j(|\xi|)-P_j^{(0)}=O(|\xi|)$ and $\mathrm{e}^{\lambda_j(|\xi|)t}\lesssim \mathrm{e}^{-c|\xi|^2t}$ for $|\xi|\to 0$, we immediately obtain the estimate

$$\big|\hat{R}_{0,1}(t,\xi)\big| \lesssim |\xi|\mathrm{e}^{-c|\xi|^2t} \quad \text{for} \quad |\xi|\to 0.$$

By a similar way, because $\lambda_j(|\xi|)-\lambda_j^0(|\xi|)=O\big(|\xi|^3\big)$ and

$$\mathrm{e}^{c|\xi|^3t}-1 = ct|\xi|^3\int_0^1 \mathrm{e}^{c|\xi|^3t\tau}\mathrm{d}\tau,$$

we have

$$\big|\hat{R}_{0,2}(t,\xi)\big| \lesssim \big|\mathrm{e}^{\lambda_j(|\xi|)t-\lambda_j^0(|\xi|)t}-1\big|\mathrm{e}^{-c|\xi|^2t} \lesssim t|\xi|^3\mathrm{e}^{-c|\xi|^2t} \lesssim |\xi|\mathrm{e}^{-c|\xi|^2t} \quad \text{for} \quad |\xi|\to 0.$$

Summarizing above estimates, we complete the proof. □

Asymptotic expansion of $\mathrm{e}^{t\hat{\Phi}(|\xi|)}$ for $|\xi|\to\infty$

From (4.15) and (4.17), we now define

$$\hat{S}_\infty(t,\xi) := \sum_{j=1}^4 \mathrm{e}^{\lambda_j^\infty(|\xi|)t}\big(P_j^{(0)}|\xi|^2+P_j^{(1)}|\xi|+P_j^{(2)}\big), \tag{4.24}$$

where

$$\begin{aligned} &\lambda_1^\infty(|\xi|) = -1-\frac{1}{b^2}|\xi|^{-2}, \quad \lambda_3^\infty(|\xi|) = -b^2|\xi|^2+1+\frac{1}{b^2}|\xi|^{-2},\\ &\lambda_2^\infty(|\xi|) = -1-\frac{1}{a^2}|\xi|^{-2}, \quad \lambda_4^\infty(|\xi|) = -a^2|\xi|^2+1+\frac{1}{a^2}|\xi|^{-2}, \end{aligned} \tag{4.25}$$

and $P_j^{(0)}$, $P_j^{(1)}$, $P_j^{(2)}$ are given in (4.18). We write the propagator for $|\xi|\to\infty$ as follows:

$$\mathrm{e}^{t\hat{\Phi}(|\xi|)} = \hat{S}_\infty(t,\xi)+\hat{R}_\infty(t,\xi). \tag{4.26}$$

Then, we can derive the following pointwise estimate for the remainder $\hat{R}_\infty(t,\xi)$ for $|\xi|\to\infty$.

Lemma 4.3.2. *We have the following estimate of remainder:*

$$\big|\hat{R}_\infty(t,\xi)\big| \lesssim \mathrm{e}^{-ct}, \tag{4.27}$$

where $|\xi|\geqslant 1/\varepsilon$ *and* c *is a positive constant.*

Proof. From (4.24) and (4.26), we obtain

$$\begin{aligned}\hat{R}_\infty(t,\xi) &= \sum_{j=1}^{4} \mathrm{e}^{\lambda_j(|\xi|)t} P_j(|\xi|) - \sum_{j=1}^{4} \mathrm{e}^{\lambda_j^\infty(|\xi|)t}\big(P_j^{(0)}|\xi|^2 + P_j^{(1)}|\xi| + P_j^{(2)}\big)\\ &= \sum_{j=1}^{4} \mathrm{e}^{\lambda_j(|\xi|)t}\big(P_j(|\xi|) - P_j^{(2)}\big) + \sum_{j=1}^{4} \mathrm{e}^{\lambda_j^\infty(|\xi|)t}\big(\mathrm{e}^{\lambda_j(|\xi|)t-\lambda_j^\infty(|\xi|)t} - 1\big)P_j^{(2)}\\ &=: \hat{R}_{\infty,1}(t,\xi) + \hat{R}_{\infty,2}(t,\xi).\end{aligned}$$

Using the facts that $P_j(|\xi|) - P_j^{(2)} = O\big(|\xi|^{-1}\big)$ and $\mathrm{e}^{\lambda_j(|\xi|)t} \lesssim \mathrm{e}^{-ct}$ for $|\xi| \to \infty$, we immediately obtain the estimate

$$\big|\hat{R}_{\infty,1}(t,\xi)\big| \lesssim |\xi|^{-1}\mathrm{e}^{-ct} \lesssim \mathrm{e}^{-ct} \quad \text{for} \quad |\xi| \to \infty.$$

Since $\lambda_j(|\xi|) - \lambda_j^\infty(|\xi|) = O\big(|\xi|^{-3}\big)$ we have

$$\big|\hat{R}_{\infty,2}(t,\xi)\big| \lesssim \big|\mathrm{e}^{\lambda_j(|\xi|)t-\lambda_j^\infty(|\xi|)t} - 1\big|\mathrm{e}^{-c|\xi|^2t} \lesssim \mathrm{e}^{-ct} \quad \text{for} \quad |\xi| \to \infty.$$

Thus, the proof is completed. □

Conclusion for asymptotic representation

Finally, combining of the estimates for $\hat{R}_0(t,\xi)$ and $\hat{R}_\infty(t,\xi)$, we can immediately prove the following statement for the asymptotic expansions of the propagator $\mathrm{e}^{t\hat{\Phi}(|\xi|)}$. The proof of the next theorem strictly follows the proof of Lemma 4.3 from the paper [39].

Theorem 4.3.1. *We have the following asymptotic expansions:*

$$\mathrm{e}^{t\hat{\Phi}(|\xi|)} = \hat{S}_0(t,\xi) + \hat{S}_\infty(t,\xi) + \hat{R}(t,\xi), \tag{4.28}$$

where the remainder $\hat{R}(t,\xi)$ satisfies the estimates

$$\big|\hat{R}(t,\xi)\big| \lesssim \begin{cases} |\xi|\mathrm{e}^{-c|\xi|^2t} & \textit{for} \quad |\xi| \leqslant \varepsilon,\\ \mathrm{e}^{-ct} & \textit{for} \quad |\xi| \geqslant \varepsilon.\end{cases}$$

4.4. Diffusion phenomena

The purpose of this section is to investigate diffusion phenomena for the Cauchy problem (4.4). According to Theorem 4.2.1 in the previous section, we observe that the decay rate of these estimates is dominated by the behavior of the eigenvalues for $|\xi| \to 0$. For frequencies in the bounded and large zones, they imply exponential decay providing that we assume a suitable regularity of initial data. Thus, we only explain diffusion phenomena of solutions for the case $|\xi| \to 0$.

To begin with, let us introduce the following reference system:

$$\begin{cases} \tilde{u}_t - \frac{1}{2}\widetilde{M}^2\Delta\tilde{u} + i\widetilde{M}(-\Delta)^{1/2}\tilde{u} = 0, & x \in \mathbb{R}^2,\ t \in \mathbb{R}_+,\\ \tilde{u}(0,x) = \tilde{u}_0(x) := \mathcal{F}^{-1}(W_0)(x), & x \in \mathbb{R}^2,\end{cases} \tag{4.29}$$

where $\tilde{u} = \big(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3, \tilde{u}^4\big)^{\mathrm{T}}$ and $\widetilde{M} := \mathrm{diag}(-b,-a,b,a)$.
This reference system consists of two different evolution equations as follows:

$$\begin{aligned} &\text{heat equation:} && \tilde{u}_t^+ - \Delta\tilde{u}^+ = 0,\\ &\text{half-wave equation:} && \tilde{u}_t^- \pm i(-\Delta)^{1/2}\tilde{u}^- = 0,\end{aligned}$$

with suitable initial data.

Remark 4.4.1. *We point out the influence of friction and Kelvin-Voigt damping to diffusion phenomena. Considering the linear dissipative elastic waves*

$$u_{tt}-a^2\Delta u-\big(b^2-a^2\big)\nabla\operatorname{div}u+u_t=0, \tag{4.30}$$

$$u_{tt}-a^2\Delta u-\big(b^2-a^2\big)\nabla\operatorname{div}u+\big(-a^2\Delta-\big(b^2-a^2\big)\nabla\operatorname{div}\big)u_t=0, \tag{4.31}$$

with $b>a>0$. From the recent papers [88, 17], we observe that the reference systems to (4.30) *can be described by heat-type systems. However, when friction is replaced by Kelvin-Voigt damping, i.e. dissipative system* (4.31)*, the reference system consists of heat systems and half-wave systems.*

Taking partial Fourier transform such that $\widetilde{W}(t,\xi)=\mathcal{F}_{x\to\xi}(\tilde{u}(t,x))$ to (4.29), we obtain

$$\begin{cases}\widetilde{W}_t+\frac{1}{2}|\xi|^2\widetilde{M}^2\widetilde{W}+i|\xi|\widetilde{M}\widetilde{W}=0, & \xi\in\mathbb{R}^2,\ t\in\mathbb{R}_+,\\ \widetilde{W}(0,\xi)=W_0(\xi), & \xi\in\mathbb{R}^2.\end{cases} \tag{4.32}$$

According to (4.20), (4.13) and (4.14), we can explicitly express $\hat{S}_0(t,\xi)$ by

$$\begin{aligned}\hat{S}_0(t,\xi)&=\operatorname{diag}\big(\mathrm{e}^{\lambda_1^{(0)}(|\xi|)t},\mathrm{e}^{\lambda_3^{(0)}(|\xi|)t},\mathrm{e}^{\lambda_2^{(0)}(|\xi|)t},\mathrm{e}^{\lambda_4^{(0)}(|\xi|)t}\big)\\ &=\operatorname{diag}\big(\mathrm{e}^{ib|\xi|t-\frac{b^2}{2}|\xi|^2t},\mathrm{e}^{ia|\xi|t-\frac{a^2}{2}|\xi|^2t},\mathrm{e}^{-ib|\xi|t-\frac{b^2}{2}|\xi|^2t},\mathrm{e}^{-ia|\xi|t-\frac{a^2}{2}|\xi|^2t}\big).\end{aligned} \tag{4.33}$$

Then, we know by direct computation that $\hat{S}_0(t,\xi)$ is the solution of the evolution system (4.32).

Considering initial data satisfying $\big(|D|u_0^k,u_1^k\big)\in L^m\big(\mathbb{R}^2\big)\times L^m\big(\mathbb{R}^2\big)$ with $m\in[1,2]$ for $k=1,2$, we state our first result.

Theorem 4.4.1. *Let us consider the Cauchy problem* (4.1)*. We assume that initial data satisfies $\big(|D|u_0^k,u_1^k\big)\in L^m\big(\mathbb{R}^2\big)\times L^m\big(\mathbb{R}^2\big)$ with $m\in[1,2]$ for $k=1,2$. Then, the following refinement estimates hold:*

$$\begin{aligned}&\big\|\chi_{\mathrm{int}}(D)\mathcal{F}^{-1}_{\xi\to x}(W-\hat{S}_0W_0)(t,\cdot)\big\|_{(\dot{H}^s(\mathbb{R}^2))^2}\\ &\lesssim(1+t)^{-\frac{2-m+ms}{2m}-\frac{1}{2}}\sum_{k=1}^{2}\big\|\big(|D|u_0^k,u_1^k\big)\big\|_{L^m(\mathbb{R}^2)\times L^m(\mathbb{R}^2)},\end{aligned}$$

where $s\geqslant 0$.

Remark 4.4.2. *If one would consider the reference system as the following type heat system only:*

$$\tilde{u}_t-\frac{1}{2}\widetilde{M}^2\Delta\tilde{u}=0,$$

or the following half-wave system only:

$$\tilde{u}_t+i\widetilde{M}(-\Delta)^{1/2}\tilde{u}=0,$$

then we cannot observe any diffusion structure. In other words, comparing Theorems 4.2.1 with 4.4.1, we observe that there is not any improvement in the decay estimates.

Proof. According to (4.22) and Lemma 4.3.1 we can estimate

$$\begin{aligned}&\big|\chi_{\mathrm{int}}(\xi)|\xi|^s(W(t,\xi)-\hat{S}_0(t,\xi)W_0(\xi))\big|\\ &=\big|\chi_{\mathrm{int}}(\xi)|\xi|^s\big(\mathrm{e}^{t\hat{\Phi}(|\xi|)}W_0(\xi)-\hat{S}_0(t,\xi)W_0(\xi)\big)\big|\\ &=\big|\chi_{\mathrm{int}}(\xi)|\xi|^s\hat{R}_0(t,\xi)W_0(\xi)\big|\\ &\lesssim\chi_{\mathrm{int}}(\xi)|\xi|^{s+1}\mathrm{e}^{-c|\xi|^2t}|W_0(\xi)|.\end{aligned} \tag{4.34}$$

Then, applying the Parseval-Plancherel theorem we derive

$$\big\|\chi_{\mathrm{int}}(D)\mathcal{F}^{-1}_{\xi\to x}(W-\hat{S}_0W_0)(t,\cdot)\big\|_{(\dot{H}^s(\mathbb{R}^2))^2}\lesssim\big\|\chi_{\mathrm{int}}(\xi)|\xi|^{s+1}\mathrm{e}^{-c|\xi|^2t}W_0(\xi)\big\|_{(L^2(\mathbb{R}^2))^2}.$$

Following the procedure of the proof of Theorem 4.2.1 we immediately complete this proof. □

Remark 4.4.3. *Comparing Theorem 4.2.1 with Theorem 4.4.1 we observe that by subtracting* $\hat{S}_0(t,\xi)W_0(\xi)$ *in the estimates, the decay rate* $(1+t)^{-\frac{1}{2}}$ *can be gained.*

4.5. Weakly coupled system of the semilinear elastic waves with Kelvin-Voigt damping in 2D

One of our goals in the last section is to develop sharp energy estimates for the linear elastic waves with Kelvin-Voigt damping in 2D with initial data

$$\big(|D|u_0^k, u_1^k\big) \in \big(H^s(\mathbb{R}^2)\cap L^m(\mathbb{R}^2)\big)\times\big(H^s(\mathbb{R}^2)\cap L^m(\mathbb{R}^2)\big)$$

for all $s\geqslant 0$ and $m\in[1,2]$. In this section we derive global (in time) existence of small data solutions to weakly coupled systems of semilinear elastic waves with Kelvin-Voigt damping in 2D by using energy estimates with initial data belonging to another function space.

Before showing energy estimates to be used in this section, we define the function spaces for all $s\geqslant 0$ and $m\in[1,2]$ such that

$$\mathcal{A}_{m,s}(\mathbb{R}^2) := \big(H^{s+1}(\mathbb{R}^2)\cap L^m(\mathbb{R}^2)\big)\times\big(H^s(\mathbb{R}^2)\cap L^m(\mathbb{R}^2)\big),$$

carrying its corresponding norm

$$\|(f,g)\|_{\mathcal{A}_{m,s}(\mathbb{R}^2)} := \|f\|_{H^{s+1}(\mathbb{R}^2)} + \|f\|_{L^m(\mathbb{R}^2)} + \|g\|_{L^s(\mathbb{R}^2)} + \|g\|_{L^m(\mathbb{R}^2)}.$$

If initial data is supposed to belong to $\mathcal{A}_{m,s}(\mathbb{R}^2)$ for $s\geqslant 0$ and $m\in[1,2]$, one can obtain the next theorem.

Theorem 4.5.1. *Let us consider the Cauchy problem* (4.1) *and initial data satisfies* $\big(u_0^k,u_1^k\big)\in\mathcal{A}_{m,s}(\mathbb{R}^2)$ *for* $k=1,2$*, where* $s\geqslant 0$ *and* $m\in[1,2]$*. Then, the following estimates hold:*

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^2)} \lesssim (1+t)^{1-\frac{2-m}{2m}}\sum_{k=1}^{2}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)}, \tag{4.35}$$

$$\big\||D|u^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^2)} + \big\|u_t^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^2)} \lesssim (1+t)^{-\frac{2-m+ms}{2m}}\sum_{k=1}^{2}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{A}_{m,s}(\mathbb{R}^2)}. \tag{4.36}$$

Proof. To get the estimate (4.36), we recall the following pointwise estimate in the Fourier space from Lemma 2.4 in the recent papers [43, 109]:

$$|\xi|^2|\hat{u}(t,\xi)|^2 + |\hat{u}_t(t,\xi)|^2 \lesssim \mathrm{e}^{-c\rho_{\varepsilon_1}(|\xi|)t}\big(|\xi|^2|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2\big),$$

where

$$\rho_{\varepsilon_1}(|\xi|) := \begin{cases} \varepsilon_1|\xi|^2 & \text{for } \ |\xi|\leqslant 1, \\ \varepsilon_1 & \text{for } \ |\xi|>1, \end{cases}$$

with $\varepsilon_1>0$. Then, the application of Hölder's inequality and the Hausdorff-Young inequality immediately implies (4.36).

To develop the estimate of the solution itself, we only need to combine the following integral formula:

$$u^k(t,x) = \int_0^t u_\tau^k(\tau,x)\mathrm{d}\tau + u_0^k(x)$$

and the estimate for $\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^2)}$ in (4.36) to complete (4.35). □

Let us consider the following weakly coupled system of semilinear elastic waves with Kelvin-Voigt damping in two dimensional space:

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + \mathbb{E}u_t = f(u), & x \in \mathbb{R}^2,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^2, \end{cases} \tag{4.37}$$

where $b > a > 0$ and the nonlinear terms on the right-hand sides are

$$f(u) := \big(|u^2|^{p_1}, |u^1|^{p_2}\big)^{\mathrm{T}}$$

with $p_1, p_2 > 1$. Using derived estimates (4.35), (4.36), Duhamel's principle and some tools in Harmonic Analysis (e.g. the Gagliardo-Nirenberg inequality), one may prove global (in time) existence of small data Sobolev solutions to (4.37).

Before stating our result for the global (in time) existence of small data energy solutions, we introduce the balanced exponent $p_{\text{bal}}(m)$ by

$$p_{\text{bal}}(m) := \frac{2(m+2)}{2-m} \quad \text{with} \quad m \in [1, 2), \tag{4.38}$$

and the balanced parameters for $k = 1, 2$,

$$\alpha_k(m) := \frac{2(1+m) + (3m+2)p_k + mp_1p_2}{2(p_1p_2 - 1)} \quad \text{with} \quad m \in [1, 2). \tag{4.39}$$

Remark 4.5.1. *We observe the relation between the balanced exponent* (4.38) *and balanced parameters* (4.39)*. For one thing, if we consider the condition $\alpha_1(m) < 1$, it also can be rewritten by*

$$p_1(p_2 + 1 - p_{\text{bal}}(m)) > p_{\text{bal}}(m).$$

For another, if we consider the condition $\alpha_2(m) < 1$, it also can be rewritten by

$$p_2(p_1 + 1 - p_{\text{bal}}(m)) > p_{\text{bal}}(m).$$

Theorem 4.5.2. *Let us assume $p_1, p_2 > 1$ and $m \in [1, 2)$, and the exponents satisfy one of the following conditions:*
(i) we assume

$$p_{\text{bal}}(m) < \min\{p_1; p_2\};$$

(ii) we assume $\alpha_1(m) < 1$ when

$$\frac{2}{m} \leqslant p_2 \leqslant p_{\text{bal}}(m) < p_1;$$

(iii) we assume $\alpha_2(m) < 1$ when

$$\frac{2}{m} \leqslant p_1 \leqslant p_{\text{bal}}(m) < p_2.$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for initial data $\big(u_0^k, u_1^k\big) \in \mathcal{A}_{m,0}\big(\mathbb{R}^2\big)$ for $k = 1, 2$, with

$$\big\|\big(u_0^1, u_1^1\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)} + \big\|\big(u_0^2, u_1^2\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)} \leqslant \varepsilon_0,$$

there is a uniquely determined energy solution

$$u \in \Big(\mathcal{C}\big([0,\infty), H^1\big(\mathbb{R}^2\big)\big) \cap \mathcal{C}^1\big([0,\infty), L^2\big(\mathbb{R}^2\big)\big)\Big)^2$$

to the Cauchy problem (4.37)*. Moreover, the following estimates hold:*

$$\begin{aligned}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^2)} &\lesssim (1+t)^{1-\frac{2-m}{2m}+\ell_k}\sum_{k=1}^{2}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)},\\ \big\||D|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^2)}+\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^2)} &\lesssim (1+t)^{-\frac{2-m}{2m}+\ell_k}\sum_{k=1}^{2}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)},\end{aligned}$$

where

$$0\leqslant \ell_k=\ell_k(m,p_k):=\begin{cases}0 & \text{if } \ p_k>p_{\mathrm{bal}}(m),\\ \epsilon_0 & \text{if } \ p_k=p_{\mathrm{bal}}(m),\\ \frac{2-m}{2m}(p_{\mathrm{bal}}(m)-p_k) & \text{if } \ p_k<p_{\mathrm{bal}}(m),\end{cases} \tag{4.40}$$

represent the (no) loss of decay in comparison with the corresponding estimates for the solution to the linear Cauchy problem (4.1) *(see Theorem 4.5.1), with $\epsilon_0>0$ being an arbitrary small constant in the limit cases that $p_k=p_{\mathrm{bal}}(m)$ for $k=1,2$.*

Remark 4.5.2. *Let us recall the weakly coupled system of semilinear damped wave equations*

$$\begin{cases}u_{tt}-\Delta u+u_t=|v|^{p_1}, & x\in\mathbb{R}^n,\ t\in\mathbb{R}_+,\\ v_{tt}-\Delta v+v_t=|u|^{p_2}, & x\in\mathbb{R}^n,\ t\in\mathbb{R}_+,\\ (u,u_t,v,v_t)(0,x)=(u_0,u_1,v_0,v_1)(x), & x\in\mathbb{R}^n,\end{cases} \tag{4.41}$$

with $n\in\mathbb{N}$ and $p_1,p_2>1$. The papers [73, 78, 104] proved the condition for the existence of global (in time) Sobolev solutions to (4.41)*, which can be described by*

$$\frac{1+\max\{p_1;p_2\}}{p_1p_2-1}<\frac{n}{2}, \quad \textit{and especially in the two dimensional case by} \quad \frac{1+\max\{p_1;p_2\}}{p_1p_2-1}<1. \tag{4.42}$$

For the Cauchy problem (4.37)*, we interpret the term*

$$\mathbb{E}u_t=\big(-a^2\Delta-\big(b^2-a^2\big)\nabla\operatorname{div}\big)u_t \quad \textit{with} \quad b>a>0,$$

as a damping term for the elastic waves. Thus, we point out the intersectional condition for the global (in time) existence of small data energy solutions to (4.37) *is*

$$\alpha_{\max}(m)=\max\{\alpha_1(m);\alpha_2(m)\}=\frac{1+m+\frac{3m+2}{2}\max\{p_1;p_2\}+\frac{m}{2}p_1p_2}{p_1p_2-1}<1.$$

Proof. First of all, by Duhamel's principle, we can reduce to consider the linear problem (4.1). In the following, we denote by $K_0=K_0(t,x)$ and $K_1=K_1(t,x)$ the fundamental solutions to the linear problem, corresponding to initial data, namely,

$$u(t,x)=K_0(t,x)\ast_{(x)}u_0(x)+K_1(t,x)\ast_{(x)}u_1(x)$$

is the solution to (4.1).

Let us define for $T>0$ the spaces of solutions $X(T)$ by

$$X(T):=\big(\mathcal{C}\big([0,T],H^1(\mathbb{R}^2)\big)\cap\mathcal{C}^1\big([0,T],L^2(\mathbb{R}^2)\big)\big)^2$$

with the corresponding norm

$$\|u\|_{X(T)}:=\sup_{t\in[0,T]}\big((1+t)^{-\ell_1}M_1\big(t;u^1\big)+(1+t)^{-\ell_2}M_2\big(t;u^2\big)\big),$$

where

$$
\begin{aligned}
M_k\big(t;u^k\big) :=&(1+t)^{-1+\frac{2-m}{2m}}\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^2)}\\
&+(1+t)^{\frac{2-m}{2m}}\Big(\big\||D|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^2)}+\big\|u_t^k(t,\cdot)\big\|_{L^2(\mathbb{R}^2)}\Big),
\end{aligned}
$$

and the parameters in the loss of decay ($\ell_k>0$) and no loss of decay ($\ell_k=0$) are defined in (4.40).

Next, we consider the integral operator $N: X(T)\to X(T)$, which is defined by

$$
Nu(t,x):=u_{\text{lin}}(t,x)+u^1_{\text{non}}(t,x)+u^2_{\text{non}}(t,x),
$$

where

$$
\begin{aligned}
u_{\text{lin}}(t,x)&=K_0(t,x)\ast_{(x)}u_0(x)+K_1(t,x)\ast_{(x)}u_1(x),\\
u^1_{\text{non}}(t,x)&=\int_0^t K_1(t-\tau,x)\ast_{(x)}|u^2(\tau,x)|^{p_1}\mathrm{d}\tau,\\
u^2_{\text{non}}(t,x)&=\int_0^t K_1(t-\tau,x)\ast_{(x)}|u^1(\tau,x)|^{p_2}\mathrm{d}\tau.
\end{aligned}
$$

From Theorem 4.5.1 and $\ell_k\geqslant 0$ for $k=1,2$, we can get the following estimates:

$$
\|u_{\text{lin}}\|_{X(T)}\lesssim\sum_{k=1}^{2}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)}.
$$

In the next step we should estimate these terms

$$
\big\|\partial_t^j|D|^lu^1_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^2)}\quad\text{and}\quad\big\|\partial_t^j|D|^lu^2_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^2)}
$$

for $j+l=0,1$ and $j,l\in\mathbb{N}_0$.
Applying the classical Gagliardo-Nirenberg inequality, we may obtain

$$
\begin{aligned}
\big\||u^2(\tau,\cdot)|^{p_1}\big\|_{L^m(\mathbb{R}^2)}&\lesssim(1+\tau)^{\left(-\frac{2-m}{2m}+\ell_2\right)p_1+\frac{2}{m}}\|u\|^{p_1}_{X(\tau)},\\
\big\||u^1(\tau,\cdot)|^{p_2}\big\|_{L^m(\mathbb{R}^2)}&\lesssim(1+\tau)^{\left(-\frac{2-m}{2m}+\ell_1\right)p_2+\frac{2}{m}}\|u\|^{p_2}_{X(\tau)},
\end{aligned}
$$

where we used our assumption $2/m\leqslant\min\{p_1;p_2\}$ for $m\in[1,2)$.
For one thing, in order to estimate u^k_{non} for $k=1,2$, we apply the derived $\big(L^2\cap L^m\big)-L^2$ estimate in $[0,t]$. For another, we use the derived $\big(L^2\cap L^m\big)-L^2$ estimate in $[0,t/2]$ and the derived L^2-L^2 estimate in $[t/2,t]$ to estimate $\partial_t^j|D|^lu^k_{\text{non}}$ for $j+l=1$ and $k=1,2$. Therefore, we obtain the following estimates for $j+l=0,1$ and $j,l\in\mathbb{N}_0$:

$$
\begin{aligned}
&(1+t)^{j+l-1+\frac{2-m}{2m}-\ell_1}\big\|\partial_t^j|D|^lu^1_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^2)}\\
&\lesssim(1+t)^{-\ell_1}\|u\|^{p_1}_{X(t)}\Big(\int_0^{t/2}(1+\tau)^{\left(-\frac{2-m}{2m}+\ell_2\right)p_1+\frac{2}{m}}\mathrm{d}\tau+(1+t)^{\left(-\frac{2-m}{2m}+\ell_2\right)p_1+\frac{2}{m}+1}\Big),\\
&(1+t)^{j+l-1+\frac{2-m}{2m}-\ell_2}\big\|\partial_t^j|D|^lu^2_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^2)}\\
&\lesssim(1+t)^{-\ell_2}\|u\|^{p_2}_{X(t)}\Big(\int_0^{t/2}(1+\tau)^{\left(-\frac{2-m}{2m}+\ell_1\right)p_2+\frac{2}{m}}\mathrm{d}\tau+(1+t)^{\left(-\frac{2-m}{2m}+\ell_1\right)p_2+\frac{2}{m}+1}\Big).
\end{aligned}
$$

We now need to distinguish between three cases. Without loss of generality, we only give the proof for the case $p_1>p_2$.

Case 1: We assume $p_{\text{bal}}(m)<\min\{p_1;p_2\}$.

In this case it allows us to assume no loss of decay, i.e., $\ell_1 = \ell_2 = 0$. Our assumption $p_{\text{bal}}(m) < \min\{p_1; p_2\}$ immediately leads to

$$-\frac{2-m}{2m}p_1 + \frac{2}{m} < -1 \quad \text{and} \quad -\frac{2-m}{2m}p_2 + \frac{2}{m} < -1.$$

Hence, we have the following estimates for $j + l = 0, 1$ and $j, l \in \mathbb{N}$:

$$(1+t)^{j+l-1+\frac{2-m}{2m}}\big(\big\|\partial_t^j|D|^l u^1_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^2)} + \big\|\partial_t^j|D|^l u^2_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^2)}\big) \lesssim \|u\|^{p_1}_{X(t)} + \|u\|^{p_2}_{X(t)}.$$

Case 2: We assume $\alpha_1(m) < 1$ if $2/m \leqslant p_2 \leqslant p_{\text{bal}}(m) < p_1$.

In this case it allows us to assume loss of decay only for the second component and its derivatives with respect to x and t, i.e., $\ell_1 = 0$ and

$$\ell_2 = \begin{cases} \epsilon_0 & \text{if } \ p_2 = p_{\text{bal}}(m), \\ \frac{2-m}{2m}(p_{\text{bal}}(m) - p_2) & \text{if } \ p_2 < p_{\text{bal}}(m). \end{cases} \tag{4.43}$$

Due to the assumption $2/m \leqslant p_2 \leqslant p_{\text{bal}}(m)$, the parameter ℓ_2 chosen in (4.43) is positive. Moreover, the assumption $\alpha_1(m) < 1$ implies that

$$1 + \frac{2}{m} - \frac{2-m}{2m}p_1 + \frac{2-m}{2m}\Big(\frac{2(m+2)}{2-m} - p_2\Big)p_1 < 0. \tag{4.44}$$

We can get these estimates from the combination of the parameter ℓ_2 chosen in (4.43), our assumptions (4.44) and $2/m \leqslant p_2 \leqslant p_{\text{bal}}(m) < p_1$

$$-\frac{2-m}{2m}p_2 + \frac{2}{m} + 1 - \ell_2 \leqslant 0 \quad \text{and} \quad \Big(-\frac{2-m}{2m} + \ell_2\Big)p_1 + \frac{2}{m} + 1 < 0.$$

Then, the following estimates hold for $j + l = 0, 1$ and $j, l \in \mathbb{N}_0$:

$$(1+t)^{j+l-1+\frac{2-m}{2m}}\big\|\partial_t^j|D|^l u^1_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|^{p_1}_{X(t)},$$
$$(1+t)^{j+l-1+\frac{2-m}{2m}-\ell_2}\big\|\partial_t^j|D|^l u^2_{\text{non}}(t,\cdot)\big\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|^{p_2}_{X(t)}.$$

Finally, combining all of the derived estimates, we can prove

$$\|Nu\|_{X(T)} \lesssim \sum_{k=1}^{2}\big\|\big(u_0^k, u_1^k\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)} + \sum_{k=1}^{2}\|u\|^{p_k}_{X(T)}, \tag{4.45}$$

uniformly with respect to $T \in [0, \infty)$.

To derive the Lipschitz condition, we can apply Hölder's inequality and the classical Gagliardo-Nirenberg inequality to get

$$\|Nu - N\bar{u}\|_{X(T)} \lesssim \|u - \bar{u}\|_{X(T)} \sum_{k=1}^{2}\big(\|u\|^{p_k-1}_{X(T)} + \|\bar{u}\|^{p_k-1}_{X(T)}\big), \tag{4.46}$$

uniformly with respect to $T \in [0, \infty)$.

These derived estimates (4.45) and (4.46) show that the mapping $N : X(T) \to X(T)$ is a contraction for initial data satisfying

$$\big\|\big(u_0^1, u_1^1\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)} + \big\|\big(u_0^2, u_1^2\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)} \leqslant \varepsilon_0,$$

with small constant $\varepsilon_0 > 0$. According to Banach's fixed-point theorem, we complete the proof. $\square$

4.6. Treatment of elastic waves with Kelvin-Voigt damping in 3D

In this section we consider the following Cauchy problem for weakly coupled systems of semilinear elastic waves with Kelvin-Voigt damping in 3D:

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2-a^2\big)\nabla \operatorname{div} u + \mathbb{E}u_t = f(u), & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ (u,u_t)(0,x) = (u_0,u_1)(x), & x\in\mathbb{R}^3, \end{cases} \tag{4.47}$$

where $b > a > 0$ and the nonlinear terms on the right-hand sides are

$$f(u) := \big(|u^3|^{p_1}, |u^1|^{p_2}, |u^2|^{p_3}\big)^{\mathrm{T}}$$

with $p_1, p_2, p_3 > 1$. For the corresponding linearized problem

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2-a^2\big)\nabla \operatorname{div} u + \mathbb{E}u_t = 0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ (u,u_t)(0,x) = (u_0,u_1)(x), & x\in\mathbb{R}^3, \end{cases} \tag{4.48}$$

it allows us to use the Helmholtz decomposition.

To begin with, let us recall the following orthogonal decomposition:

$$\big(L^2(\mathbb{R}^3)\big)^3 = \overline{\nabla H^1(\mathbb{R}^3)} \oplus \mathcal{D}_0(\mathbb{R}^3),$$

where the space $\overline{\nabla H^1(\mathbb{R}^3)}$ denotes the vector fields with divergence zero and $\mathcal{D}_0(\mathbb{R}^3)$ denotes the vector fields with curl zero (c.f. [60]).
Thus, we can decompose the solution $u = u(t,x)$ to the linearized problem (4.48) into a potential and a solenoidal part

$$u = u^{p_0} \oplus u^{s_0},$$

where the vector unknown $u^{p_0} = u^{p_0}(t,x)$ stands for rotation-free and the vector unknown $u^{s_0} = u^{s_0}(t,x)$ stands for divergence-free in a weak sense.

Taking account of the relation

$$\nabla \operatorname{div} u = \nabla\times(\nabla\times u) + \Delta u$$

in three dimensions, we can decouple the system (4.48) into two viscoelastic damped wave equations with different propagation speeds a as well as b, respectively,

$$\begin{cases} u^{s_0}_{tt} - a^2\Delta u^{s_0} - a^2\Delta u^{s_0}_t = 0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ \big(u^{s_0},u^{s_0}_t\big)(0,x) = \big(u^{s_0}_0,u^{s_0}_1\big)(x), & x\in\mathbb{R}^3, \end{cases} \tag{4.49}$$

and

$$\begin{cases} u^{p_0}_{tt} - b^2\Delta u^{p_0} - b^2\Delta u^{p_0}_t = 0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ \big(u^{p_0},u^{p_0}_t\big)(0,x) = \big(u^{p_0}_0,u^{p_0}_1\big)(x), & x\in\mathbb{R}^3. \end{cases} \tag{4.50}$$

The well-posedness of weak solutions to (4.49) and (4.50) has been studied in [47], and the well-posedness of distributional solutions has been investigated in [25]. Some $L^2 - L^2$ estimates and $\big(L^2\cap L^m\big) - L^2$ estimates of solutions with $m \in [1,2)$ also have been developed in [25, 24]. Furthermore, $L^p - L^q$ estimates not necessarily on the conjugate line of solution to the Cauchy problems (4.49) or (4.50) have been investigated in [84, 94]. Lastly, we mention that asymptotic profiles of solutions with initial data is taken from weighted L^1 spaces have been studied in [40, 69, 70].

To study the Cauchy problem (4.47), we next derive $\big(L^2\cap L^m\big) - L^2$ estimates and $L^2 - L^2$ estimates of solutions to the linearized Cauchy problem (4.48). According to the paper [109], one can obtain the next estimates.

Theorem 4.6.1. *Let us consider the Cauchy problem* (4.48) *and initial data satisfies* $\big(u_0^k,u_1^k\big)\in\mathcal{A}_{m,s}\big(\mathbb{R}^3\big)$ *for* $k=1,2,3$, *where* $s\geqslant 0$ *and* $m\in[1,2]$. *Then, the following estimates hold:*

$$\big\|u^k(t,\cdot)\big\|_{L^2(\mathbb{R}^3)}\lesssim(1+t)^{-\frac{6-5m}{4m}}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^3)}\ \ \textit{if}\ \ m\in\big[1,\tfrac{6}{5}\big),$$

$$\big\||D|u^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}+\big\|u_t^k(t,\cdot)\big\|_{\dot{H}^s(\mathbb{R}^3)}\lesssim(1+t)^{-\frac{6-3m+2sm}{4m}}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{\mathcal{A}_{m,s}(\mathbb{R}^3)}.$$

Now, we state our theorem for the global (in time) existence of small data energy solution to (4.47). To begin with, let us introduce the balanced parameter $\tilde{p}_{\text{bal}}(m)$, $\tilde{\alpha}_1(m)$ and $\tilde{\tilde{\alpha}}_1(m)$ for $m\in[1,6/5)$ by

$$\tilde{p}_{\text{bal}}(m):=\frac{3+2m}{3-m}, \tag{4.51}$$

$$\tilde{\alpha}_1(m):=\frac{m(2+3p_2+p_1p_2)}{2(p_1p_2-1)}, \tag{4.52}$$

$$\tilde{\tilde{\alpha}}_1(m):=\frac{m(2+3(p_2+1)p_3+p_1p_2p_3)}{2(p_1p_2p_3-1)}. \tag{4.53}$$

Remark 4.6.1. *Here we point out the relation between these parameters. If we consider the condition* $\tilde{\alpha}_1(m)<3/2$, *it also can be rewritten by*

$$p_2(p_1+1-\tilde{p}_{\text{bal}}(m))>\tilde{p}_{\text{bal}}(m).$$

If we consider the condition $\tilde{\tilde{\alpha}}_1(m)<3/2$, *it also can be rewritten by*

$$p_3(p_2(p_1+1-\tilde{p}_{\text{bal}}(m))+1-\tilde{p}_{\text{bal}}(m))>\tilde{p}_{\text{bal}}(m).$$

Remark 4.6.2. *From the recent paper [17], we remark that the balanced exponent shown in* (4.51) *and the balanced parameters shown in* (4.52) *as well as* (4.53) *correspond to the balanced parameters to the weakly coupled system of the semilinear viscoelastic damped elastic waves in 3D.*

One can follow the procedure of the proof of Theorem 5.5 in [17] to obtain the following theorem. Without loss of generality, we assume $p_1<p_2<p_3$.

Theorem 4.6.2. *Let us assume* $1<p_1<p_2<p_3$ *and* $m\in[1,6/5)$, *and the exponents satisfy one of the following conditions:*
(i) we assume

$$\tilde{p}_{\text{bal}}(m)<p_1<p_2<p_3\leqslant 3;$$

(ii) we assume $\tilde{\alpha}_1(m)<3/2$ *when*

$$\frac{2}{m}\leqslant p_1\leqslant\tilde{p}_{\text{bal}}(m)<p_2<p_3\leqslant 3;$$

(iii) we assume $\tilde{\tilde{\alpha}}_1(m)<3/2$ *when*

$$\frac{2}{m}\leqslant p_1<p_2\leqslant\tilde{p}_{\text{bal}}(m)<p_3\leqslant 3.$$

Then, there exists a constant $\varepsilon_0>0$ *such that for initial data* $\big(u_0^k,u_1^k\big)\in\mathcal{A}_{m,0}\big(\mathbb{R}^3\big)$ *for* $k=1,2,3$, *with*

$$\big\|\big(u_0^1,u_1^1\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^3)}+\big\|\big(u_0^2,u_1^2\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^3)}+\big\|\big(u_0^3,u_1^3\big)\big\|_{\mathcal{A}_{m,0}(\mathbb{R}^3)}\leqslant\varepsilon_0,$$

there is a uniquely determined energy solution

$$u\in\Big(\mathcal{C}\big([0,\infty),H^1\big(\mathbb{R}^3\big)\big)\cap\mathcal{C}^1\big([0,\infty),L^2\big(\mathbb{R}^3\big)\big)\Big)^3$$

to the Cauchy problem (4.47). *Moreover, the following estimates hold:*

$$\left\|u^k(t,\cdot)\right\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{6-5m}{4m}+\tilde{\ell}_k} \sum_{k=1}^{3} \left\|\left(u_0^k, u_1^k\right)\right\|_{\mathcal{A}_{m,0}(\mathbb{R}^3)},$$

$$\left\||D|u^k(t,\cdot)\right\|_{L^2(\mathbb{R}^3)} + \left\|u_t^k(t,\cdot)\right\|_{L^2(\mathbb{R}^3)} \lesssim (1+t)^{-\frac{6-3m}{4m}+\tilde{\ell}_k} \sum_{k=1}^{3} \left\|\left(u_0^k, u_1^k\right)\right\|_{\mathcal{A}_{m,0}(\mathbb{R}^3)},$$

where

$$0 \leqslant \tilde{\ell}_1 = \tilde{\ell}_1(m,p_1) := \begin{cases} 0 & \text{if } p_1 > \tilde{p}_{\text{bal}}(m), \\ \epsilon_0 & \text{if } p_1 = \tilde{p}_{\text{bal}}(m), \\ \frac{3-m}{2m}(\tilde{p}_{\text{bal}}(m) - p_1) & \text{if } p_1 < \tilde{p}_{\text{bal}}(m), \end{cases}$$

$$0 \leqslant \tilde{\ell}_2 = \tilde{\ell}_2(m,p_2) := \begin{cases} 0 & \text{if } p_2 > \tilde{p}_{\text{bal}}(m), \\ \epsilon_0 & \text{if } p_2 = \tilde{p}_{\text{bal}}(m), \\ \frac{3-m}{2m}((\tilde{p}_{\text{bal}}(m) - p_1)p_2 + (\tilde{p}_{\text{bal}}(m) - p_2)) & \text{if } p_2 < \tilde{p}_{\text{bal}}(m), \end{cases}$$

and $\tilde{\ell}_3 = 0$, *represent the (no) loss of decay in comparison with the corresponding estimates for the solution to the Cauchy problem* (4.48) *(see Theorem 4.6.1), with* $\epsilon_0 > 0$ *being an arbitrary small constant in the limit cases that* $p_k = \tilde{p}_{\text{bal}}(m)$ *for* $k = 1, 2, 3$.

4.7. Concluding remarks

Remark 4.7.1. *One may derive asymptotic profiles for elastic waves with Kelvin-Voigt damping in 3D, that is the model* (4.48). *Before doing this, one may apply a diagonalization procedure (e.g. [17]), or asymptotic expansions of eigenvalues and their eigenprojections (e.g. the method used in Section 4.3, or [39]) to get representations of solutions in the Fourier space. Basing on this representations, one may derive asymptotic profiles in a framework of weighted* L^1 *data by using the tools developed in [40].*

Remark 4.7.2. *In this chapter we only study qualitative properties of solutions to the linear Cauchy problem in the* L^2 *norm. For estimates of solutions in the* L^q *norm and diffusion phenomena in* $L^p - L^q$ *framework, where* $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, *we may apply the theory developed in Subsection 2.6.2.*

Remark 4.7.3. *In Section 4.5 we have proved the global (in time) existence of small data energy solutions to the Cauchy problem* (4.37). *One also can prove the global (in time) existence of small data Sobolev solutions*

$$u \in \left(\mathcal{C}\left([0,\infty), H^{s+1}\left(\mathbb{R}^2\right)\right) \cap \mathcal{C}^1\left([0,\infty), H^{s}\left(\mathbb{R}^2\right)\right)\right)^2$$

to the Cauchy problem (4.37) *with initial data taking from* $\mathcal{A}_{m,s}\left(\mathbb{R}^2\right)$ *for* $s > 0$ *and* $m \in [1,2)$ *by following the next strategy.*

For higher regular data and even not embedded in $L^\infty\left(\mathbb{R}^2\right)$ *(i.e.* $0 < s < 1$*), we can apply the fractional Gagliardo-Nirenberg inequality, the fractional chain rule, the fractional Leibniz rule (c.f. [92, 35, 32, 81]). More precisely, we apply the fractional chain rule to get an estimate for the nonlinear term in Riesz potential spaces* $\dot{H}^s\left(\mathbb{R}^2\right)$ *with* $s \in (0,1)$. *For example,*

$$\left\||u^2(\tau,\cdot)|^{p_1}\right\|_{\dot{H}^s(\mathbb{R}^2)} \lesssim \left\|u^2(\tau,\cdot)\right\|^{p_1-1}_{L^{q_1}(\mathbb{R}^2)} \left\|u^2(\tau,\cdot)\right\|_{\dot{H}^s_{q_2}(\mathbb{R}^2)},$$

where $\frac{p_1-1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$ *and* $p_1 > \lceil s \rceil$. *Here,* $\lceil \cdot \rceil$ *denotes the smallest integer large than a given number,* $\lceil s \rceil := \min\left\{\tilde{s} \in \mathbb{Z} : s \leqslant \tilde{s}\right\}$. *Then, one can estimate the terms on the right-hand sides by the*

fractional Gagliardo-Nirenberg inequality.
To estimate the difference between the nonlinearities, we set

$$g\big(u^2\big) = u^2|u^2|^{p_1-2}$$

to get

$$|u^2(\tau,x)|^{p_1} - |\tilde{u}^2(\tau,x)|^{p_1} = p_1 \int_0^1 \big(u^2(\tau,x) - \tilde{u}^2(\tau,x)\big) g\big(\nu u^2(\tau,x) + (1-\nu)\tilde{u}^2(\tau,x)\big)\mathrm{d}\nu.$$

Then, applying the fractional Leibniz rule we obtain

$$\begin{aligned}
&\big\||u^2(\tau,\cdot)|^{p_1} - |\tilde{u}^2(\tau,\cdot)|^{p_1}\big\|_{\dot{H}^s(\mathbb{R}^2)} \\
&\lesssim \int_0^1 \big\|u^2(\tau,\cdot) - \tilde{u}^2(\tau,\cdot)\big\|_{\dot{H}^s_{r_1}(\mathbb{R}^2)} \big\|g\big(\nu u^2(\tau,\cdot) + (1-\nu)\tilde{u}^2(\tau,\cdot)\big)\big\|_{L^{r_2}(\mathbb{R}^2)}\mathrm{d}\nu \\
&\quad + \int_0^1 \big\|u^2(\tau,\cdot) - \tilde{u}^2(\tau,\cdot)\big\|_{L^{r_3}(\mathbb{R}^2)} \big\|g\big(\nu u^2(\tau,\cdot) + (1-\nu)\tilde{u}^2(\tau,\cdot)\big)\big\|_{\dot{H}^s_{r_4}(\mathbb{R}^2)}\mathrm{d}\nu,
\end{aligned}$$

where $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4} = \frac{1}{2}$. *We next use the fractional Gagliardo-Nirenberg inequality again to estimate all terms on the right-hand side. Thus, after choosing suitable parameters* $q_1, q_2, r_1, r_2, r_3, r_4$, *a new lower bound* $1 + \lceil s \rceil$ *for the exponent* p_1 *appears.*

For large regular initial data with $s > 1$, *it allows us to use the fractional powers (c.f. [23]) and the continuous embedding* $H^s\big(\mathbb{R}^2\big) \hookrightarrow L^\infty\big(\mathbb{R}^2\big)$. *At this time, we need to give a new lower bound* $1 + s$ *for the exponents.*

Remark 4.7.4. *In Theorem 4.6.2 we only show the global existence result for the energy solution to* (4.47) *with initial data belonging to* $\mathcal{A}_{m,0}\big(\mathbb{R}^3\big)$ *for* $m \in [1, 6/5)$. *If one is interested in initial data is taken from* $\mathcal{A}_{m,s}\big(\mathbb{R}^3\big)$ *for all* $m \in [1, 2)$ *and* $s \geqslant 0$, *one can read Section 5 of the recent paper [17].*

5.Linear elastic waves in 3D

5.1. Introduction

In this chapter we derive some qualitative properties of solutions to linear elastic waves with vanishing right-hand sides, which are modeled by

$$\begin{cases} u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u = 0, & x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^3, \end{cases} \tag{5.1}$$

where the unknown $u = \big(u^1, u^2, u^3\big)^{\mathrm{T}} \in \mathbb{R}^3$ denotes the elastic displacement, t stands for the time and x stands for the space-variables. The constants a and b are related to the Lamé constants satisfying $b > a > 0$.

Our main purpose of this chapter is to study qualitative properties of solutions to linear elastic waves in 3D. Particularly, we investigate some properties for the solution itself. To do this, we should derive representations of solutions in the physical space and in the Fourier space, respectively. For one thing, to derive $L^p - L^q$ estimates away of the conjugate line, our main tool is the boundedness of the Fourier integral operator associated with some interpolation theorems. For another, to derive estimates of radial solutions, we may reduce three dimensional elastic waves to solutions to Euler-Poisson-Darboux equations and estimate their solutions.

The rest of the chapter is organized as follows. In Section 5.2 by deriving representations of solutions, we study H^s well-posedness of the Cauchy problem and finite propagation speed of the Sobolev solutions to the Cauchy problem (5.1). Then, we investigate $L^p - L^q$ estimates away of the conjugate line by interpolation between $L^p - L^q$ estimates on the conjugate line and $L^p - L^p$ estimates with $p \in (1, \infty)$ in Section 5.3. In Section 5.4 we study estimates of radial solutions to the Cauchy problem (5.1). In the last section an introduction of an open problem for semilinear elastic waves in 3D completes the chapter.

5.2. Qualitative properties of solutions

In this section we investigate H^s well-posedness for the Cauchy problem (5.1) by using representations of solutions in the Fourier space. Then, by applying explicit formulas of solutions, the finite propagation speed of solutions is derived. In other words, our first step is to derive representations of solutions to the Cauchy problem (5.1).

Motivated by the pioneering paper [1], we may derive representations of solution by transferring the Cauchy problem for linear elastic waves to the Cauchy problem for fourth-order equations (see (5.4) later). First of all, applying the operator div to the Cauchy problem (5.1) implies

$$\begin{cases} (\operatorname{div} u)_{tt} - b^2\Delta(\operatorname{div} u) = 0, & x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ (\operatorname{div} u, \operatorname{div} u_t)(0, x) = (\operatorname{div} u_0, \operatorname{div} u_1)(x), & x \in \mathbb{R}^3. \end{cases} \tag{5.2}$$

Next, we act the scalar operator $\partial_t^2 - b^2\Delta$ on equations in (5.1) to find

$$\begin{aligned} 0 &= \big(\partial_t^2 - b^2\Delta\big)\big(\partial_t^2 u - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u\big) \\ &= u_{tttt} - \big(a^2 + b^2\big)\Delta u_{tt} + a^2b^2\Delta^2 u + \big(b^2 - a^2\big)\big(b^2\Delta\nabla \operatorname{div} u - \nabla \operatorname{div} u_{tt}\big). \end{aligned} \tag{5.3}$$

Lastly, we plug the equation of (5.2) into (5.3) to find that each component $u^j = u^j(t, x)$ with $j = 1, 2, 3$ of the solution $u = u(t, x)$ to the Cauchy problem (5.1) satisfies the following Cauchy problem for scaler equation of fourth-order:

$$\begin{cases} u^j_{tttt} - \big(a^2 + b^2\big)\Delta u^j_{tt} + a^2b^2\Delta^2 u^j = 0, & x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ \big(u^j, u^j_t, u^j_{tt}, u^j_{ttt}\big)(0, x) = \big(u^j_0, u^j_1, u^j_2, u^j_3\big)(x), & x \in \mathbb{R}^3, \end{cases} \tag{5.4}$$

where $b > a > 0$ and initial data is represented by

$$u_2^j(x) = a^2 \Delta u_0^j(x) + \big(b^2 - a^2\big)\partial_{x_j} \operatorname{div} u_0(x),$$
$$u_3^j(x) = a^2 \Delta u_1^j(x) + \big(b^2 - a^2\big)\partial_{x_j} \operatorname{div} u_1(x),$$

where $j = 1, 2, 3$.

Applying the partial Fourier transform with respect to spatial variables to (5.4), i.e., $\hat{u}^j(t,\xi) = \mathcal{F}_{x\to\xi}\big(u^j(t,x)\big)$ for $j = 1, 2, 3$, we obtain the following ordinary differential equation depending on the parameter $|\xi|$:

$$\begin{cases} \hat{u}^j_{tttt} + \big(a^2+b^2\big)|\xi|^2 \hat{u}^j_{tt} + a^2b^2|\xi|^4 \hat{u}^j = 0, & \xi \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ \big(\hat{u}^j, \hat{u}^j_t, \hat{u}^j_{tt}, \hat{u}^j_{ttt}\big)(0,\xi) = \big(\hat{u}^j_0, \hat{u}^j_1, \hat{u}^j_2, \hat{u}^j_3\big)(\xi), & \xi \in \mathbb{R}^3, \end{cases} \tag{5.5}$$

where initial data is represented by

$$\begin{aligned} \hat{u}_2^j(\xi) &= -a^2|\xi|^2\hat{u}_0^j(\xi) - \big(b^2-a^2\big)\xi_j(\xi\cdot\hat{u}_0(\xi)), \\ \hat{u}_3^j(\xi) &= -a^2|\xi|^2\hat{u}_1^j(\xi) - \big(b^2-a^2\big)\xi_j(\xi\cdot\hat{u}_1(\xi)). \end{aligned} \tag{5.6}$$

Here dot $\cdot$ denotes the usual inner product in $\mathbb{R}^3$.

Equation (5.4) is strictly hyperbolic, with the symbol

$$\begin{aligned} P(\lambda, i\xi) &= \lambda^4 + \big(a^2+b^2\big)|\xi|^2\lambda^2 + a^2b^2|\xi|^4 \\ &= \big(\lambda^2 + a^2|\xi|^2\big)\big(\lambda^2 + b^2|\xi|^2\big). \end{aligned}$$

The characteristic roots of (5.5) are

$$\lambda_{1,2}(|\xi|) = \pm ia|\xi| \quad \text{and} \quad \lambda_{3,4}(|\xi|) = \pm ib|\xi|.$$

Therefore, by using (5.6), the solution to (5.5) is given by

$$\begin{aligned} \hat{u}^j(t,\xi) &= \frac{b^2\cos(a|\xi|t) - a^2\cos(b|\xi|t)}{b^2-a^2}\hat{u}_0^j(\xi) + \Big(\frac{b^2\sin(a|\xi|t)}{a(b^2-a^2)|\xi|} - \frac{a^2\sin(b|\xi|t)}{b(b^2-a^2)|\xi|}\Big)\hat{u}_1^j(\xi) \\ &\quad + \frac{\cos(a|\xi|t) - \cos(b|\xi|t)}{(b^2-a^2)|\xi|^2}\hat{u}_2^j(\xi) + \Big(\frac{\sin(a|\xi|t)}{a(b^2-a^2)|\xi|^3} - \frac{\sin(b|\xi|t)}{b(b^2-a^2)|\xi|^3}\Big)\hat{u}_3^j(\xi) \\ &= \cos(a|\xi|t)\hat{u}_0^j(\xi) - (\cos(a|\xi|t) - \cos(b|\xi|t))\frac{\xi_j}{|\xi|^2}\sum_{k=1}^3 \xi_k \hat{u}_0^k(\xi) \\ &\quad + \frac{\sin(a|\xi|t)}{a|\xi|}\hat{u}_1^j(\xi) - \Big(\frac{\sin(a|\xi|t)}{a|\xi|} - \frac{\sin(b|\xi|t)}{b|\xi|}\Big)\frac{\xi_j}{|\xi|^2}\sum_{k=1}^3 \xi_k\hat{u}_1^k(\xi), \end{aligned} \tag{5.7}$$

where $j = 1, 2, 3$. In other words, we now have obtained a representation of solution in the Fourier space.

Employing the representation of solution (5.7), the H^s well-posedness of the Cauchy problem (5.1) is investigated by the following theorem.

Theorem 5.2.1. *Let us consider the Cauchy problem* (5.1) *and initial data satisfies* $\big(u_0^k, u_1^k\big) \in H^{s+1}\big(\mathbb{R}^3\big) \times H^s\big(\mathbb{R}^3\big)$ *for* $k = 1, 2, 3$ *with* $s \geqslant 0$*. Then, there exists a uniquely determined energy solution of higher-order*

$$u \in \Big(\mathcal{C}\big([0,\infty), H^{s+1}\big(\mathbb{R}^3\big)\big) \cap \mathcal{C}^1\big([0,\infty), H^s\big(\mathbb{R}^3\big)\big)\Big)^3.$$

Remark 5.2.1. *One also can derive* H^s *well-posedness of distributional solutions even for all* $s \in \mathbb{R}$ *by using a similar approach of the proof of Theorem 5.2.1.*

Proof. For one thing, the uniqueness of solution to the Cauchy problem (5.1) can be obtained by using the fact that the total energy to (5.1) is conserved, which can be expressed by

$$\begin{aligned}&\frac{1}{2}\Big(\|u_t(t,\cdot)\|^2_{(L^2(\mathbb{R}^3))^3}+a^2\|\nabla\times u(t,\cdot)\|^2_{(L^2(\mathbb{R}^3))^3}+b^2\|\operatorname{div}u(t,\cdot)\|^2_{(L^2(\mathbb{R}^3))^3}\Big)\\&=\frac{1}{2}\Big(\|u_1\|^2_{(L^2(\mathbb{R}^3))^2}+a^2\|\nabla\times u_0\|^2_{(L^2(\mathbb{R}^3))^3}+b^2\|\operatorname{div}u_0\|^2_{(L^2(\mathbb{R}^3))^3}\Big).\end{aligned}$$

For another, the existence of energy solutions of higher-order can be derived by following the same approach of proving the existence of energy solutions to the Cauchy problem for the wave equation (e.g. Chapter 14 in [25]). Here we only need to use the representation of solution (5.7) and the next inequality for $k=1,2,3$:

$$\int_{\mathbb{R}^3}\langle\xi\rangle^{2s}\Big|\frac{\xi_j}{|\xi|^2}\sum_{k=1}^3\xi_k\hat{u}_l^k(\xi)\Big|^2\mathrm{d}\xi\lesssim\sum_{k=1}^3\int_{\mathbb{R}^3}\langle\xi\rangle^{2s}\big|\hat{u}_l^k(\xi)\big|^2\mathrm{d}\xi,$$

for all $l=0,1$. Then, the proof is complete. □

In order to derive Sobolev solutions with finite propagation speed to the Cauchy problem (5.1), we now investigate explicit formulas of solutions to (5.4). The main idea is strongly motived by [51, 1]. From (5.4), we observe that each component $u^j=u^j(t,x)$ with $j=1,2,3$, of the solution $u=u(t,x)$ satisfies

$$\begin{cases}\big(\partial_t^2-a^2\Delta\big)\big(\partial_t^2-b^2\Delta\big)u^j=0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ \big(u^j,u^j_t,u^j_{tt},u^j_{ttt}\big)(0,x)=\big(u^j_0,u^j_1,u^j_2,u^j_3\big)(x), & x\in\mathbb{R}^3,\end{cases}\tag{5.8}$$

which can be solved successively first for $w^j:=\big(\partial_t^2-b^2\Delta\big)u^j$ and then for u^j.
Obviously, $w^j=w^j(t,x)$ with $j=1,2,3$, is the solution to the Cauchy problem for the free wave equation

$$\begin{cases}w^j_{tt}-a^2\Delta w^j=0, & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ \big(w^j,w^j_t\big)(0,x)=\big(w^j_0,w^j_1\big)(x), & x\in\mathbb{R}^3,\end{cases}\tag{5.9}$$

where

$$\begin{aligned}w^j_0(x)&:=\big(b^2-a^2\big)\big(\partial_{x_j}\operatorname{div}u_0(x)-\Delta u^j_0(x)\big),\\ w^j_1(x)&:=\big(b^2-a^2\big)\big(\partial_{x_j}\operatorname{div}u_1(x)-\Delta u^j_1(x)\big).\end{aligned}$$

By applying Kirchhoff's formula, the solution to (5.9) is explicitly expressed by

$$\begin{aligned}w^j(t,x)=&\frac{\partial}{\partial t}\Big(\frac{(b^2-a^2)t}{4\pi}\int_{S^2}\big(\partial_{x_j}\operatorname{div}u_0-\Delta u^j_0\big)(x+at\omega)\mathrm{d}\omega\Big)\\&+\frac{(b^2-a^2)t}{4\pi}\int_{S^2}\big(\partial_{x_j}\operatorname{div}u_1-\Delta u^j_1\big)(x+at\omega)\mathrm{d}\omega,\end{aligned}$$

where $S^2=\{\omega=(\omega_1,\omega_2,\omega_3):\ |\omega|=1\}$ with its surface element $\mathrm{d}\omega$.
Next, the solution $u^j=u^j(t,x)$ to the inhomogeneous wave equation

$$\begin{cases}u^j_{tt}-b^2\Delta u^j=w^j(t,x), & x\in\mathbb{R}^3,\ t\in\mathbb{R}_+,\\ \big(u^j,u^j_t\big)(0,x)=\big(u^j_0,u^j_1\big)(x), & x\in\mathbb{R}^3,\end{cases}$$

is given by

$$\begin{aligned}u^j(t,x)=&\frac{\partial}{\partial t}\Big(\frac{t}{4\pi}\int_{S^2}u^j_0(x+bt\omega)d\omega\Big)+\frac{t}{4\pi}\int_{S^2}u^j_1(x+bt\omega)\mathrm{d}\omega\\&+\frac{b^2-a^2}{(4\pi)^2}\int_0^t(t-\tau)\frac{\partial}{\partial\tau}\Big(\tau\int_{S^2}\int_{\tilde{S}^2}\big(\partial_{x_j}\operatorname{div}u_0-\Delta u^j_0\big)(x+b(t-\tau)\tilde{\omega}+a\tau\omega)d\tilde{\omega}\mathrm{d}\omega\Big)\mathrm{d}\tau\\&+\frac{b^2-a^2}{(4\pi)^2}\int_0^t(t-\tau)\tau\int_{S^2}\int_{\tilde{S}^2}\big(\partial_{x_j}\operatorname{div}u_1-\Delta u^j_1\big)(x+b(t-\tau)\tilde{\omega}+a\tau\omega)\mathrm{d}\tilde{\omega}\mathrm{d}\omega\mathrm{d}\tau.\end{aligned}$$

Then, according to the paper [51], one may derive the explicit formula of solution for $j = 1, 2, 3$,

$$\begin{aligned} u^j(t,x) = \frac{1}{4\pi}\sum_{k=1}^{3}\Big[\int_{r=at} & r^{-2}\Big(t(\delta_{jk}-\eta_j\eta_k)\Big(u_1^k(y)+a\frac{\mathrm{d}}{\mathrm{d}\mathbf{n}}u_0^k(y)\Big)+(2\delta_{jk}-4\eta_j\eta_k)u_0^k(y)\Big)\mathrm{d}S_y \\ &+\int_{r=bt} r^{-2}\Big(t\eta_j\eta_k\Big(u_1^k(y)+b\frac{\mathrm{d}}{\mathrm{d}\mathbf{n}}u_0^k(y)\Big)+(-\delta_{jk}+4\eta_j\eta_k)u_0^k(y)\Big)\mathrm{d}S_y \\ &-\int_{at<r<bt} r^{-3}(\delta_{jk}-3\eta_j\eta_k)\big(u_0^k(y)+tu_1^k(y)\big)\mathrm{d}y\Big], \end{aligned}$$

with the Kronecker delta δ_{jk}, where $r = |y-x|$, $\eta_j = r^{-1}(y_j - x_j)$ and $\mathbf{n}$ is the exterior unit normal. With the help of the above representation, we may prove the next result.

Theorem 5.2.2. *Let us consider the Cauchy problem* (5.1) *and initial data satisfies* $\big(u_0^k, u_1^k\big) \in H^1\big(\mathbb{R}^3\big)\times L^2\big(\mathbb{R}^3\big)$ *having their support such that* $\operatorname{supp} u_0^k,\ u_1^k \subset \{|x|\leqslant R\}$ *for* $k = 1, 2, 3$. *Then, the solution*

$$u \in \big(\mathcal{C}\big([0,\infty), L^4\big(\mathbb{R}^3\big)\big)\big)^3,$$

to the Cauchy problem (5.1) *having its support such that*

$$\operatorname{supp} u(t,\cdot) \subset (\{at-R \leqslant |x| \leqslant bt+R\})^3.$$

Proof. From our assumption that initial data has its support such that $\operatorname{supp} u_0^k,\ u_1^k \subset \{|x|\leqslant R\}$ for $k = 1, 2, 3$, we can obtain the support of the solution by applying the explicit formula of solution to the Cauchy problem (5.1). Then, applying $H^1\big(\mathbb{R}^3\big) \subset L^4\big(\mathbb{R}^3\big)$ by the Sobolev embedding theorem to Theorem 5.2.1, we complete the proof of the theorem. □

5.3. $L^p - L^q$ estimates of solutions

It is well-known that to establish $L^p - L^q$ estimates is the crucial step in studying the local/global (in time) existence of solutions to nonlinear Cauchy problems. In this section we derive $L^p - L^q$ estimates for a pair of exponents $(1/p, 1/q)$ locating in the triangle region with vertices $P_1 = (3/4, 1/4)$, $P_2 = (0,0)$ and $P_3 = (1,1)$ (see Figure 5.1).

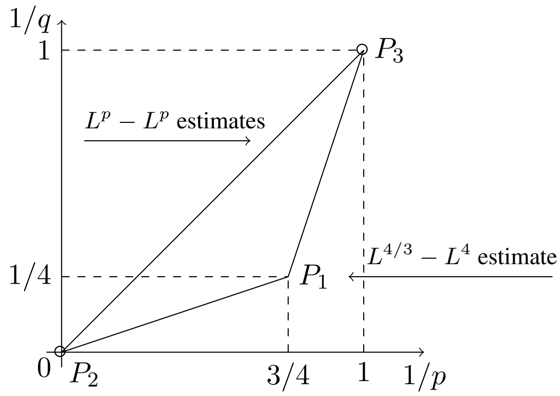


Fig. 5.1.: Admissible range of a pair of exponents $(1/p, 1/q)$

To do this, we mainly use representations of solutions in the Fourier space (5.7). Furthermore, similar as the proof of $L^p - L^q$ estimates for the free wave equation (e.g. Chapter 16 in [25]), we divide the proof into three steps. First of all, we derive $L^p - L^q$ estimates on the conjugate line, i.e., $1/p + 1/q = 1$. Next, we derive $L^p - L^p$ estimates for $p \in (1,\infty)$. Finally, combining with derived estimates in previous two steps, we obtain $L^p - L^q$ estimates away of the conjugate line by applying interpolation theorem.

Let us state the result of $L^p - L^q$ estimates on the conjugate line.

Theorem 5.3.1. *Let us consider the Cauchy problem* (5.1) *and initial data satisfies* $\big(u_0^k, u_1^k\big) \in \dot{H}_p^1\big(\mathbb{R}^3\big) \times L^p\big(\mathbb{R}^3\big)$ *for* $k = 1, 2, 3$. *Then, the solution satisfies the following singular* $L^p - L^q$ *estimates for any* $t > 0$*:*

$$\big\|u^j(t,\cdot)\big\|_{L^q(\mathbb{R}^3)} \lesssim t^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{\dot{H}_p^1(\mathbb{R}^3)\times L^p(\mathbb{R}^3)},$$

with $j = 1, 2, 3$, *where* $1 < p \leqslant 2 \leqslant q < \infty$, $2(1/p - 1/q) \leqslant 1 \leqslant 3(1/p - 1/q)$ *and* $1/p + 1/q = 1$.

Remark 5.3.1. *If one is interested in* $L^p - L^q$ *estimates on the conjugate line without singularity for* $t \to +0$, *we may consider initial data taking from higher-order Bessel potential spaces such that*

$$\big\|u^j(t,\cdot)\big\|_{L^q(\mathbb{R}^3)} \lesssim (1+t)^{-\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{k=1}^{3} \big\|\big(u_0^k, u_1^k\big)\big\|_{H_p^{M_p}(\mathbb{R}^3)\times H_p^{M_p-1}(\mathbb{R}^3)},$$

with $j = 1, 2, 3$, *where* $1 < p \leqslant 2$, $1/p + 1/q = 1$, *and the real number* M_p *satisfies* $M_p \geqslant 3(1/p - 1/q) \geqslant 1$.

Remark 5.3.2. *We observe that the rate of estimates and data spaces in* $L^p - L^q$ *estimates for elastic waves stated in Theorem 5.3.1 and Remark 5.3.1 are the same as* $L^p - L^q$ *estimates for solutions to the free wave equation.*

Proof. Let us rewrite the representations of solution (5.7) by

$$\begin{aligned}
u^j(t,x) =& \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{2}\big(\mathrm{e}^{ia|\xi|t} + \mathrm{e}^{-ia|\xi|t}\big)\Big) *_{(x)} u_0^j(x) + \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{2i|\xi|}\big(\mathrm{e}^{ia|\xi|t} - \mathrm{e}^{-ia|\xi|t}\big)\Big) *_{(x)} u_1^j(x) \\
&+ \sum_{k=1}^{3} \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{2}\frac{\xi_j\xi_k}{|\xi|^2}\big(\big(\mathrm{e}^{ib|\xi|t} + \mathrm{e}^{-ib|\xi|t}\big) - \big(\mathrm{e}^{ia|\xi|t} + \mathrm{e}^{-ia|\xi|t}\big)\big)\Big) *_{(x)} u_0^k(x) \\
&+ \sum_{k=1}^{3} \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{2i|\xi|}\frac{\xi_j\xi_k}{|\xi|^2}\big(\big(\mathrm{e}^{ib|\xi|t} - \mathrm{e}^{-ib|\xi|t}\big) - \big(\mathrm{e}^{ia|\xi|t} - \mathrm{e}^{-ia|\xi|t}\big)\big)\Big) *_{(x)} u_1^k(x),
\end{aligned}$$

where $j = 1, 2, 3$.

To derive $L^p - L^q$ estimates on the conjugate line, we need to estimate the Fourier multipliers in the above representation, that is,

$$I_1(t,x;\kappa) := \mathcal{F}^{-1}_{\xi\to x}\Big(\mathrm{e}^{-ic|\xi|t}\frac{1}{|\xi|^{2\kappa}}\Big), \tag{5.10}$$

$$I_2(t,x;\kappa) := \mathcal{F}^{-1}_{\xi\to x}\Big(\mathrm{e}^{-ic|\xi|t}\frac{\xi_j\xi_k}{|\xi|^2}\frac{1}{|\xi|^{2\kappa}}\Big), \tag{5.11}$$

for $j, k = 1, 2, 3$, where $\kappa \geqslant 0$, $c > 0$.

Let us decompose the extended phase space $(0,\infty) \times \mathbb{R}^3_\xi$ into

the pseudo-differential zone:	$\mathcal{Z}_{\text{pd}} = \big\{(t,\xi) \in (0,\infty)\times\mathbb{R}^3_\xi : t|\xi| \leqslant 1\big\}$,
the hyperbolic zone	$\mathcal{Z}_{\text{hyp}} = \big\{(t,\xi) \in (0,\infty)\times\mathbb{R}^3_\xi : t|\xi| \geqslant 1\big\}$,

with the cutting function $\chi \in \mathcal{C}^\infty\big(\mathbb{R}^3_\xi\big)$ satisfying $\chi(\xi) \equiv 0$ for $|\xi| \leqslant 1/2$, $\chi(\xi) \equiv 1$ for $|\xi| \geqslant 3/4$, and $\chi(\xi) \in [0,1]$. In the following parts, we estimate the Fourier multipliers (5.10) and (5.11) in the pseudo-differential zone and the hyperbolic zone, respectively.

To estimate the Fourier multiplier $I_1(t,x;\kappa)$ we refer the reader to [5, 91]. There the authors proved

$$t^{-2\kappa+3\left(\frac{1}{p}-\frac{1}{q}\right)} I_1(t,\cdot;\kappa) \in L_p^q\big(\mathbb{R}^3\big), \tag{5.12}$$

for $1<p\leqslant 2\leqslant q<\infty$, $2(1/p-1/q)\leqslant 2\kappa\leqslant 3(1/p-1/q)$ and $1/p+1/q=1$.
Then, we may derive the next estimates by plugging $\kappa=1/2$ into (5.12):

$$\Big\|\mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{2|\xi|}\big(\mathrm{e}^{ia|\xi|t}+\mathrm{e}^{-ia|\xi|t}\big)\Big)*_{(x)}|D|u_0^j(x)\Big\|_{L^q(\mathbb{R}^3)}\lesssim t^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)}\|u_0^j\|_{\dot{H}^1_p(\mathbb{R}^3)}, \tag{5.13}$$

$$\Big\|\mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{2i|\xi|}\big(\mathrm{e}^{ia|\xi|t}-\mathrm{e}^{-ia|\xi|t}\big)\Big)*_{(x)}u_1^j(x)\Big\|_{L^q(\mathbb{R}^3)}\lesssim t^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)}\|u_1^j\|_{L^p(\mathbb{R}^3)}, \tag{5.14}$$

with $j=1,2,3$, where $1<p\leqslant 2\leqslant q<\infty$, $2(1/p-1/q)\leqslant 1\leqslant 3(1/p-1/q)$ and $1/p+1/q=1$.

Next, we derive estimates of the Fourier multiplier $I_2(t,x;\kappa)$ only. To do this, we divide the proof in to three steps.

Step 1: We estimate the Fourier multiplier $I_2(t,x;\kappa)$ in the pseudo-differential zone.

We apply the change of variables $\eta:=t\xi$ and $tz:=x$ to get

$$\begin{aligned} &t^{-2\kappa+3\left(\frac{1}{p}-\frac{1}{q}\right)}\|I_2(t,x;\kappa)*_{(x)}(1-\chi(t|D|))\|_{L^q(\mathbb{R}^3)}\\ &=\Big\|\mathcal{F}^{-1}_{\eta\to z}\Big(\mathrm{e}^{-ic|\eta|}\frac{\eta_j\eta_k}{|\eta|^2}\frac{1-\chi(|\eta|)}{|\eta|^{2\kappa}}\Big)\Big\|_{L^q(\mathbb{R}^3)}. \end{aligned} \tag{5.15}$$

Due to the oscillating integral satisfying the estimate

$$\operatorname{meas}\Big\{\eta\in\mathbb{R}^3:\Big|\mathrm{e}^{-ic|\eta|}\frac{\eta_j\eta_k}{|\eta|^2}\frac{1-\chi(|\eta|)}{|\eta|^{2\kappa}}\Big|\geqslant\ell\Big\}\leqslant\operatorname{meas}\big\{\eta\in\mathbb{R}^3:|\eta|\leqslant\ell^{-\frac{1}{2\kappa}}\big\}\lesssim\ell^{-\frac{3}{2\kappa}},$$

by applying Proposition B.3.1, the Fourier multiplier satisfies

$$\mathcal{F}^{-1}_{\eta\to z}\Big(\mathrm{e}^{-ic|\eta|}\frac{\eta_j\eta_k}{|\eta|^2}\frac{1-\chi(|\eta|)}{|\eta|^{2\kappa}}\Big)\in L^q_p(\mathbb{R}^3) \tag{5.16}$$

for $1<p\leqslant 2\leqslant q<\infty$ and $0\leqslant 2\kappa\leqslant 3(1/p-1/q)$.

Step 1.2: We estimate the Fourier multiplier $I_2(t,x;\kappa)$ in the hyperbolic zone.

First of all, we introduce a dyadic decomposition, where a cutting function $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^3)$ with the compact support $\{\xi\in\mathbb{R}^3:|\xi|\in[1/2,2]\}$ is chosen. Moreover, we define

$$\varphi_l(\xi):=\varphi(2^{-l}\xi)\quad\text{for}\quad l\geqslant 1,\quad\text{and}\quad\varphi_0(\xi):=1-\sum_{l=1}^{\infty}\varphi_l(\xi).$$

We know for $l\leqslant l_0$ with a large constant l_0 that the following estimate holds:

$$\|I_2(t,x;\kappa)*_{(x)}(\chi(t|D|)\varphi_l(t|D|))\|_{L^\infty(\mathbb{R}^3)}\lesssim t^{2\kappa-3},$$

because we have the following estimate for $l\leqslant l_0$:

$$\begin{aligned} &\Big\|\mathcal{F}^{-1}_{\xi\to x}\Big(\mathrm{e}^{-ic|\xi|t}\frac{\xi_j\xi_k}{|\xi|^2}\frac{\chi(t|\xi|)\varphi_l(t|\xi|)}{|\xi|^{2\kappa}}\Big)\Big\|_{L^\infty(\mathbb{R}^3)}\\ &\lesssim\Big|\int_{2^{l-1}t^{-1}}^{2^{l+1}t^{-1}}\nu^{2-2\kappa}\mathrm{d}\nu\Big|=2^{(3-2\kappa)l}\big(2^{3-2\kappa}-2^{-3+2\kappa}\big)t^{2\kappa-3}. \end{aligned}$$

To get the estimate for $l>l_0$, we apply the ansatz $t\xi=2^l\eta$ and Young's inequality to get

$$\|I_2(t,x)*_{(x)}(\chi(t|D|)\varphi_l(t|D|))\|_{L^\infty(\mathbb{R}^3)}=2^{l(3-2\kappa)}t^{2\kappa-3}\Big\|\mathcal{F}^{-1}_{\eta\to x}\Big(\mathrm{e}^{-ic2^l|\eta|}\frac{\eta_j\eta_k}{|\eta|^2}\frac{\varphi(|\eta|)}{|\eta|^{2\kappa}}\Big)\Big\|_{L^\infty(\mathbb{R}^3)}.$$

Then, a Littman type lemma (c.f. with Proposition B.4.1) shows that

$$\Big\|\mathcal{F}^{-1}_{\eta\to x}\Big(\mathrm{e}^{-ic2^l|\eta|}\frac{\eta_j\eta_k}{|\eta|^2}\frac{\varphi(|\eta|)}{|\eta|^{2\kappa}}\Big)\Big\|_{L^\infty(\mathbb{R}^3)} \lesssim 2^{-l}.$$

Hence, we conclude the following $L^1 - L^\infty$ estimate:

$$\|I_2(t,x;\kappa) *_{(x)} (\chi(t|D|)\varphi_l(t|D|))\|_{L^\infty(\mathbb{R}^3)} \lesssim 2^{2l(1-\kappa)}t^{2\kappa-3}.$$

For the $L^2 - L^2$ estimate, we apply the Parseval-Plancherel theorem to get

$$\|I_2(t,x;\kappa) *_{(x)} (\chi(t|D|)\varphi_l(t|D|))\|_{L^2(\mathbb{R}^3)} \lesssim 2^{-2l\kappa}t^{2\kappa}.$$

By the Riesz-Thorin interpolation theorem we conclude

$$t^{-2\kappa+3\left(\frac{1}{p}-\frac{1}{q}\right)} I_2(t,x;\kappa) *_{(x)} \chi(t|D|) \in L^q_p\big(\mathbb{R}^3\big). \tag{5.17}$$

where $2\kappa \leqslant 2(1/p - 1/q)$ and $1/p + 1/q = 1$ with $1 < p \leqslant 2$.

Step 1.3: We summarize the above derived estimates for the Fourier multiplier.

Summarizing above estimates (5.15), (5.16), (5.17) and choosing $\kappa = 1/2$ lead to

$$t^{-1+3\left(\frac{1}{p}-\frac{1}{q}\right)} I_2(t,\cdot;\kappa) \in L^q_p\big(\mathbb{R}^3\big), \tag{5.18}$$

for $1 < p \leqslant 2 \leqslant q < \infty$, $2(1/p - 1/q) \leqslant 1 \leqslant 3\big(1/p - 1/q\big)$ and $1/p + 1/q = 1$.
In other words, the next estimates hold for $j, k = 1, 2, 3$:

$$\begin{aligned}
&\Big\|\mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{2}\frac{\xi_j\xi_k}{|\xi|^2}\frac{1}{|\xi|}\big(\big(\mathrm{e}^{ib|\xi|t}+\mathrm{e}^{-ib|\xi|t}\big)-\big(\mathrm{e}^{ia|\xi|t}+\mathrm{e}^{-ia|\xi|t}\big)\big)\Big) *_{(x)} |D|u_0^k(x)\Big\|_{L^q(\mathbb{R}^3)}\\
&\quad+\Big\|\mathcal{F}^{-1}_{\xi\to x}\Big(\frac{1}{2i|\xi|}\frac{\xi_j\xi_k}{|\xi|^2}\big(\big(\mathrm{e}^{ib|\xi|t}-\mathrm{e}^{-ib|\xi|t}\big)-\big(\mathrm{e}^{ia|\xi|t}-\mathrm{e}^{-ia|\xi|t}\big)\big)\Big) *_{(x)} u_1^k(x)\Big\|_{L^q(\mathbb{R}^3)}\\
&\lesssim t^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)}\big\|\big(u_0^k,u_1^k\big)\big\|_{\dot{H}^1_p(\mathbb{R}^3)\times L^p(\mathbb{R}^3)}.
\end{aligned}$$

Then, using (5.18) in the representations of solutions and combining with (5.13) as well as with (5.14), the proof of the theorem is completed. □

Corollary 5.3.1. *Let us consider the Cauchy problem* (5.1) *and initial data satisfies* $\big(u_0^k, u_1^k\big) \in H^{M_p}_p\big(\mathbb{R}^3\big) \times H^{M_p-1}_p\big(\mathbb{R}^3\big)$ *for* $k = 1, 2, 3$. *Then, the derivatives of solutions satisfy the following* $L^p - L^q$ *estimates for any* $t > 0$*:*

$$\big\|u_t^j(t,\cdot)\big\|_{L^q(\mathbb{R}^3)} + \big\||D|u^j(t,\cdot)\big\|_{L^q(\mathbb{R}^3)} \lesssim (1+t)^{-\left(\frac{1}{p}-\frac{1}{q}\right)}\sum_{k=1}^{3}\big\|\big(u_0^k,u_1^k\big)\big\|_{H^{M_p}_p(\mathbb{R}^3)\times H^{M_p-1}_p(\mathbb{R}^3)},$$

with $j = 1, 2, 3$, *where* $1 < p \leqslant 2$, $1/p + 1/q = 1$ *and the real number* M_p *fulfills* $M_p \geqslant 3(1/p - 1/q) + 1$.

Proof. The proof strictly follows the proof of Theorem 5.3.1. □

Secondly, we state our result on $L^p - L^p$ estimates with $p \in (1, \infty)$.

Theorem 5.3.2. *Let us consider the Cauchy problem* (5.1) *and initial data satisfies* $\big(u_0^k, u_1^k\big) \in H^1_p\big(\mathbb{R}^3\big) \times L^p\big(\mathbb{R}^3\big)$ *with* $p \in (1,\infty)$ *for* $k = 1, 2, 3$. *Then, the solution satisfies the following singular* $L^p - L^p$ *estimates for any* $t > 0$*:*

$$\big\|u^j(t,\cdot)\big\|_{H^s_p(\mathbb{R}^3)} \lesssim \max\{1;t\}\sum_{k=1}^{3}\big\|u_0^k\big\|_{H^1_p(\mathbb{R}^3)} + t\sum_{k=1}^{3}\big\|u_1^k\big\|_{L^p(\mathbb{R}^3)}, \tag{5.19}$$

with $j = 1, 2, 3$, *where* $s = 1 - 2|1/p - 1/2|$.

Remark 5.3.3. *Comparing the $L^p - L^p$ estimates for elastic waves stated in Theorem 5.3.2 with those for solutions to the free wave equation shown in [102, 82], we observe that in the case when $p \in (1,\infty)$, the time-dependent coefficient of the above estimates and data spaces are the same. Nevertheless, we did not get estimates for elastic waves in the cases when $p = 1$ and $p = \infty$. One may find the reason in Remark 5.3.4 in detail.*

Remark 5.3.4. *The restriction $p \in (1,\infty)$ in Theorem 5.3.2 is due to the application of the boundededness of the operator of the Riesz transform from $L^p(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ if $p \in (1,\infty)$. However, in the limit cases ($p = 1,\infty$), the Riesz transforms are not bounded anymore.*

Proof. From the representations of solution in (5.7), we may rewrite $u^j = u^j(t,x)$ for $j = 1,2,3$, by

$$\begin{aligned} u^j(t,x) =& \mathcal{F}^{-1}_{\xi\to x}(\cos(a|\xi|t)) *_{(x)} u_0^j(x) + \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{\sin(a|\xi|t)}{a|\xi|}\Big) *_{(x)} u_1^j(x) \\ &+ \sum_{k=1}^{3} \mathcal{F}^{-1}_{\xi\to x}(\cos(a|\xi|t) - \cos(b|\xi|t)) *_{(x)} R_{jk}u_0^k(x) \\ &+ \sum_{k=1}^{3} \mathcal{F}^{-1}_{\xi\to x}\Big(\frac{\sin(a|\xi|t)}{a|\xi|} - \frac{\sin(b|\xi|t)}{b|\xi|}\Big) *_{(x)} R_{jk}u_1^k(x), \end{aligned} \tag{5.20}$$

where R_{jk} denotes the second-order Riesz transform, which is given by a Fourier multiplier as follows:

$$\mathcal{F}(R_{jk}\phi)(\xi) := -\frac{\xi_j\xi_k}{|\xi|^2}\mathcal{F}(\phi)(\xi) \quad \text{for} \quad j,k = 1,2,3.$$

To get the desired $L^p - L^p$ estimates, we need to estimate all Fourier multipliers in (5.20).

Let us define operators S_α for α in the strip $0 \leqslant \text{Re } \alpha \leqslant (n+1)/2$, by

$$\mathcal{F}(S_\alpha\phi)(\xi) = |\xi|^{\alpha-\frac{n}{2}} J_{\frac{n}{2}-\alpha}(|\xi|)\hat{\phi}(\xi),$$

where $J_{\frac{n}{2}-\alpha}(|\xi|)$ stands for the Bessel function. Especially, taking $\alpha = (n-1)/2$ and $\alpha = (n+1)/2$, the explicit formulas of the corresponding Bessel functions are given by

$$J_{\frac{1}{2}}(|\xi|) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}|\xi|^{-\frac{1}{2}}\sin(|\xi|) \quad \text{and} \quad J_{-\frac{1}{2}}(|\xi|) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}|\xi|^{-\frac{1}{2}}\cos(|\xi|),$$

respectively. Hence, the solution can be represented by using Bessel functions as follows:

$$\begin{aligned} u^j(t,x) =& \Big(\frac{\pi}{2}\Big)^{\frac{1}{2}}\delta(at)S_2\delta\big((at)^{-1}\big)u_0^j(x) + \Big(\frac{\pi}{2}\Big)^{\frac{1}{2}}t\delta(at)S_1\delta\big((at)^{-1}\big)u_1^j(x) \\ &+ \Big(\frac{\pi}{2}\Big)^{\frac{1}{2}}\sum_{k=1}^{3}\big(\delta(at)S_2\delta\big((at)^{-1}\big)\big(R_{jk}u_0^k(x)\big) - \delta(bt)S_2\delta\big((bt)^{-1}\big)\big(R_{jk}u_0^k(x)\big)\big) \\ &+ \Big(\frac{\pi}{2}\Big)^{\frac{1}{2}}t\sum_{k=1}^{3}\big(\delta(at)S_1\delta\big((at)^{-1}\big)\big(R_{jk}u_1^k(x)\big) - \delta(bt)S_1\delta\big((bt)^{-1}\big)\big(R_{jk}u_1^k(x)\big)\big), \end{aligned} \tag{5.21}$$

with $j = 1,2,3$, where $\delta(t)$ is the dilation operator such that $\delta(t)\phi(x) = \phi(tx)$.

Form the papers [102] and [82], the following estimates hold:

$$\big\|\delta(at)S_1\delta\big((at)^{-1}\big)\phi\big\|_{H_p^s(\mathbb{R}^3)} \lesssim \|\phi\|_{L^p(\mathbb{R}^3)}, \tag{5.22}$$

$$\big\|\delta(at)S_2\delta\big((at)^{-1}\big)\phi\big\|_{H_p^s(\mathbb{R}^3)} \lesssim \max\{1;t\}\|\phi\|_{H_p^1(\mathbb{R}^3)}, \tag{5.23}$$

where $|1/p - 1/2| \leqslant 1/2$ and $s = 1 - 2|1/p - 1/2|$.
Moreover, according to the paper [48], we know that for any function $g \in L^p\big(\mathbb{R}^3\big)$ with $p \in (1, \infty)$ that the second-order Riesz transform operator is bounded such that

$$\|R_{jk}g\|_{L^p(\mathbb{R}^3)} \leqslant E_p^2 \|g\|_{L^p(\mathbb{R}^3)}, \tag{5.24}$$

with a positive constant E_p with respect to p (see [48, 4]), where $j, k = 1, 2, 3$.

Thus, applying (5.22), (5.23) and (5.24) in (5.21) we obtain

$$\begin{aligned} \big\|u^j(t,\cdot)\big\|_{H_p^s(\mathbb{R}^3)} &\lesssim \max\{1;t\}\Big(\big\|u_0^j\big\|_{H_p^1(\mathbb{R}^3)} + \sum_{k=1}^{3}\big\|R_{jk}u_0^k\big\|_{H_p^1(\mathbb{R}^3)}\Big) \\ &\quad + t\Big(\big\|u_1^j\big\|_{L^p(\mathbb{R}^3)} + \sum_{k=1}^{3}\big\|R_{jk}u_1^k\big\|_{L^p(\mathbb{R}^3)}\Big) \\ &\lesssim \max\{1;t\}\sum_{k=1}^{3}\big\|u_0^k\big\|_{H_p^1(\mathbb{R}^3)} + t\sum_{k=1}^{3}\big\|u_1^k\big\|_{L^p(\mathbb{R}^3)} \end{aligned}$$

for $s = 1 - 2|1/p - 1/2|$ and $p \in (1, \infty)$, where $j = 1, 2, 3$. It completes the proof. □

Corollary 5.3.2. *Let us consider the Cauchy problem* (5.1) *and initial data satisfies* $\big(u_0^k, u_1^k\big) \in H_p^1\big(\mathbb{R}^3\big) \times L^p\big(\mathbb{R}^3\big)$ *with* $p \in (2/(2-\epsilon_0), 2/\epsilon_0)$ *carrying* $\epsilon_0 \in (0,1)$ *for* $k = 1, 2, 3$*. Then, the solution satisfies the following singular* $L^p - L^p$ *estimates for any* $t > 0$*:*

$$\big\|u^j(t,\cdot)\big\|_{L^p(\mathbb{R}^3)} \lesssim \max\{1;t\}\sum_{k=1}^{3}\big\|u_0^k\big\|_{H_p^1(\mathbb{R}^3)} + t\sum_{k=1}^{3}\big\|u_1^k\big\|_{L^p(\mathbb{R}^3)}, \tag{5.25}$$

with $j = 1, 2, 3$.

Remark 5.3.5. *If we take* $\epsilon_0 \to +0$ *in Corollary 5.3.2, then we can obtain the largest admissible range of values for p, that is,* $p \in (1, \infty)$.

Proof. Choosing $p = 2/\epsilon_0$ and $p = 2/(2-\epsilon_0)$ with $\epsilon_0 \in (0,1)$ in Theorem 5.3.2, we get the following $L^p - L^p$ estimates for $j = 1, 2, 3$:

$$\big\|u^j(t,\cdot)\big\|_{H_p^{\epsilon_0}(\mathbb{R}^3)} \lesssim \max\{1;t\}\sum_{k=1}^{3}\big\|u_0^k\big\|_{H_p^1(\mathbb{R}^3)} + t\sum_{k=1}^{3}\big\|u_1^k\big\|_{L^p(\mathbb{R}^3)}. \tag{5.26}$$

From Sobolev embedding, we know $H_p^{\epsilon_0}\big(\mathbb{R}^3\big) \hookrightarrow L^p\big(\mathbb{R}^3\big)$ for $p \in (1, \infty)$ and $\epsilon_0 p < 3$. Then, by interpolation theorem, we obtain that (5.26) holds for all

$$\frac{2}{2-\epsilon_0} < p < \frac{2}{\epsilon_0} < \frac{3}{\epsilon_0}, \quad \text{where} \quad \epsilon_0 \in (0,1).$$

Thus, the proof is completed. □

Theorem 5.3.3. *Let us consider the Cauchy problem* (5.1) *and initial data satisfies*

$$\Big(|D|^{2\left(\frac{1}{p}-\frac{1}{q}\right)}u_0^k, u_1^k\Big) \in H_p^{1-2\left(\frac{1}{p}-\frac{1}{q}\right)}\big(\mathbb{R}^3\big) \times L^p\big(\mathbb{R}^3\big)$$

for $k = 1, 2, 3$*. Then, the solution satisfies the following* $L^p - L^q$ *estimates for* $j = 1, 2, 3$*, and any* $t > 0$*:*

$$\begin{aligned} \big\|u^j(t,\cdot)\big\|_{L^q(\mathbb{R}^3)} &\lesssim \max\Big\{t^{-\left(\frac{1}{p}-\frac{1}{q}\right)}; t^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)}\Big\}\sum_{k=1}^{3}\Big\||D|^{2\left(\frac{1}{p}-\frac{1}{q}\right)}\langle D\rangle^{1-2\left(\frac{1}{p}-\frac{1}{q}\right)}u_0^k\Big\|_{L^p(\mathbb{R}^3)} \\ &\quad + t^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)}\sum_{k=1}^{3}\big\|u_1^k\big\|_{L^p(\mathbb{R}^3)}, \end{aligned} \tag{5.27}$$

where every pair of exponents $(1/p, 1/q)$ *belongs to the triangle region* $\overline{P_1P_2P_3}$ *(see Figure 5.2) with vertices* $P_1 = (3/4, 1/4)$*,* $P_2 = (0, 0)$ *and* $P_3 = (1, 1)$*.*

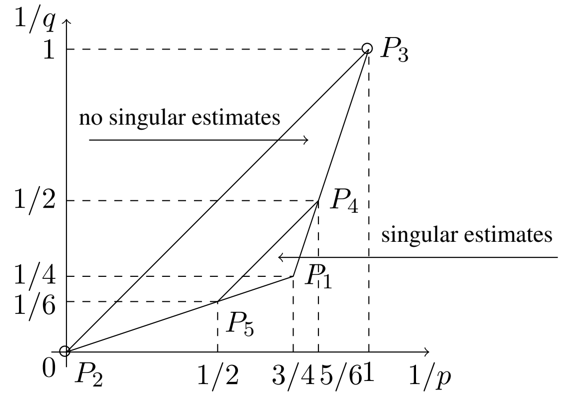


Fig. 5.2.: Admissible range of a pair of exponents $(1/p, 1/q)$

Remark 5.3.6. *The singularity for* $t \to +0$ *of the estimate in Theorem 5.3.3 will disappear if we choose vanishing first data* $u_0 \equiv 0$ *and* $1 - 3(1/p - 1/q) \geqslant 0$*. In other words, the following estimates hold for* $j = 1, 2, 3$*:*

$$\left\|u^j(t,\cdot)\right\|_{L^q(\mathbb{R}^3)} \lesssim t^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{k=1}^{3} \left\|u_1^k\right\|_{L^p(\mathbb{R}^3)}, \tag{5.28}$$

where every pair of exponents $(1/p, 1/q)$ *belongs to the trapezium region* $\overline{P_2P_3P_4P_5}$ *with vertices* $P_2 = (0, 0)$*,* $P_3 = (1, 1)$*,* $P_4 = (5/6, 1/2)$ *and* $P_5 = (1/2, 1/6)$*. However, if we take* $u_0 \not\equiv 0$*, to obtain no singular estimates for* $t \to +0$*, we need to assume* (p, q) *belonging to the following set*

$$\left\{(p,q):\ -\left(\tfrac{1}{p}-\tfrac{1}{q}\right) \geqslant 0,\quad 1-3\left(\tfrac{1}{p}-\tfrac{1}{q}\right) \geqslant 0 \ \ \text{and} \ \ p \leqslant q\right\} = \{(p,q):\ p = q\}.$$

It means in the case when $u_0 \not\equiv 0$*, no singular estimates for* $t \to +0$ *hold only if we assume* $p = q \in (1, \infty)$ *in* (5.27)*.*

Proof. From Theorem 5.3.1, we may choose $1/p = 3/4$ and $1/q = 1/4$ to get

$$\left\|u^j(t,\cdot)\right\|_{L^4(\mathbb{R}^3)} \lesssim t^{-\frac{1}{2}} \sum_{k=1}^{3} \left\||D|u_0^k\right\|_{L^{4/3}(\mathbb{R}^3)} + t^{-\frac{1}{2}} \sum_{k=1}^{3} \left\|u_1^k\right\|_{L^{4/3}(\mathbb{R}^3)}, \tag{5.29}$$

with $j = 1, 2, 3$. Then, taking $\epsilon_0 \to +0$ in (5.25) and applying the Riesz-Thorin interpolation theorem between (5.25) and (5.29) complete the proof. □

5.4. Estimates for radial solutions

In this section we are going to study estimates for radial solutions to the Cauchy problem (5.1). To begin with, let us introduce the definition of radial functions in a vector sense.

Definition 5.4.1 (see [50])**.** *A vector function* $f = f(x)$ *is called radial if it has the form* $f(x) = xg(r)$*,* $r = |x|$*, where* $g = g(r)$ *is a scalar radial function.*

Let us consider the Cauchy problem (5.1) with radial data, i.e.,

$$u_0(x) = xv_0(r) \ \text{ and } \ u_1(x) = xv_1(r). \tag{5.30}$$

In this part we consider radial solutions, which are given by

$$u(t,x) = xv(t,r). \tag{5.31}$$

Due to the fact that

$$\frac{\partial u^j(t,x)}{\partial x_k} = \frac{x_j x_k}{|x|} v_r(t,r) \quad \text{and} \quad \frac{\partial u^k(t,x)}{\partial x_j} = \frac{x_j x_k}{|x|} v_r(t,r),$$

we obtain

$$\nabla \times u = \left(\frac{\partial u^3}{\partial x_2} - \frac{\partial u^2}{\partial x_3}, \frac{\partial u^1}{\partial x_3} - \frac{\partial u^3}{\partial x_1}, \frac{\partial u^2}{\partial x_1} - \frac{\partial u^1}{\partial x_2}\right) = 0.$$

According to the following relation in 3D:

$$\nabla \times (\nabla \times u) = -\Delta u + \nabla \operatorname{div} u,$$

we know that $u = u(t,x)$ fulfilling (5.31) is the solution to

$$\begin{cases} u_{tt} - b^2 \Delta u = 0, & x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbb{R}^3, \end{cases} \tag{5.32}$$

where initial data satisfies (5.30).
In other words, the solution $v = v(t,r)$ fulfills the following linear wave equation:

$$\begin{cases} v_{tt} - b^2 v_{rr} - \frac{4}{r} b^2 v_r = 0, & r \in \mathbb{R}_+,\ t \in \mathbb{R}_+, \\ (v, v_t)(0,r) = (v_0, v_1)(r), & r \in \mathbb{R}_+. \end{cases} \tag{5.33}$$

According to Lemma 3.1 in [59], the solution to (5.33) is explicitly given by

$$v(t,r) = \frac{1}{b}\frac{\partial}{\partial t}\Big(\int_{|bt-r|}^{bt+r} v_0(\lambda) K(\lambda,t,r)\mathrm{d}\lambda\Big) + \frac{1}{b}\int_{|bt-r|}^{bt+r} v_1(\lambda) K(\lambda,t,r)\mathrm{d}\lambda, \tag{5.34}$$

where

$$K(\lambda,t,r) = \frac{1}{2}\Big(\frac{\lambda}{r}\Big)^3 \Big(\frac{\partial}{\partial \lambda}\frac{1}{2\lambda}\Big)\big((\lambda - (bt-r))(\lambda - (bt+r))\big).$$

Furthermore, from Lemma 2.3 in [59], we have the following result: if $|bt - r| \leqslant \lambda \leqslant bt + r$, then it holds

$$|K(\lambda,t,r)| \lesssim r^{-2}\lambda^2,$$

and if $bt \leqslant 2r$, then it holds

$$|\partial_r K(\lambda,t,r)| \lesssim r^{-2}\lambda.$$

Actually, some a-priori estimates for solutions to the linear homogeneous Cauchy problem (5.33) have been investigated in [58]. According to the paper [58], we have the following two lemmas.

Lemma 5.4.1. *Let $v = v(t,r)$ be the solution to the linear Cauchy problem* (5.33) *defined by* (5.34). *Let $(t,r) \in [0,\infty) \times (0,\infty)$ be such that $bt \geqslant 2r$. Then, we have for $\ell = 0,1$*

$$\big|\partial_r^\ell v(t,r)\big| \lesssim r^{-1-\ell}\Big(\int_{bt-r}^{bt+r} \big|(\lambda v_0(\lambda))' + \big(\lambda^2 v_0(\lambda)\big)''\big|\mathrm{d}\lambda + \frac{1}{b}\int_{bt-r}^{bt+r} \big|\lambda v_1(\lambda) - \big(\lambda^2 v_1(\lambda)\big)'\big|\mathrm{d}\lambda\Big).$$

Lemma 5.4.2. *Let $v = v(t,r)$ be the solution to the linear Cauchy problem* (5.33) *defined by* (5.34). *Let $(t,r) \in [0,\infty) \times (0,\infty)$ be such that $bt \leqslant 2r$. Then, we have*

$$|v(t,r)| \lesssim r^{-2} \int_{|bt-r|}^{bt+r} \Big(\lambda |v_0(\lambda)| + \frac{1}{b}\lambda^2 |v_1(\lambda)|\Big) \mathrm{d}\lambda + \sum_{\pm} r^{-2} |bt \pm r|^2 |v_0(|bt \pm r|)|,$$

$$\begin{aligned}|\partial_r v(t,r)| \lesssim{}& r^{-2} \int_{|bt-r|}^{bt+r} \Big(|v_0(r)| + \frac{1}{b}\lambda |v_1(r)|\Big) \mathrm{d}\lambda + \sum_{\pm} r^{-2} |bt \pm r| |v_0(|bt \pm r|)| \\ &+ \sum_{\pm} r^{-2} |bt \pm r|^2 \Big(|v_0'(|bt \pm r|) + \frac{1}{b}|v_1(|bt \pm r|)|\Big).\end{aligned}$$

Finally, let us consider initial data satisfying

$$\begin{aligned}&\big|\partial_t^{\ell} v_0(r)\big| \leqslant \varepsilon_0 (1+r)^{-(\kappa+\ell)} && \text{for } \ \ell = 0,1,2, \\ &\big|\partial_t^{\ell} v_1(r)\big| \leqslant \varepsilon_0 (1+r)^{-(\kappa+1+\ell)} && \text{for } \ \ell = 0,1,\end{aligned} \tag{5.35}$$

where ε_0, κ are positive parameters to be determined later. Then, Proposition 2.3 in [58] derived some decay estimates for the solution to the linear Cauchy problem (5.34).

Proposition 5.4.1. *Let $v = v(t,r)$ be the solution to the linear Cauchy problem* (5.33) *defined by* (5.34), *with $v_0(r) \in \mathcal{C}^2((0,\infty))$, $v_1(r) \in \mathcal{C}^1((0,\infty))$ satisfying* (5.35) *for some $\kappa > 2$ and $\varepsilon_0 > 0$. Then, we have for any $(t,r) \in [0,\infty) \times (0,\infty)$*

$$\begin{aligned}|v(t,r)| &\leqslant C_0 \varepsilon_0 r (1+r)^{-1} \psi_\kappa(t,r), \\ |\partial_r v(t,r)| &\leqslant C_0 \varepsilon_0 r^{-1} \psi_\kappa(t,r),\end{aligned}$$

where C_0 is a positive constant that is independent of t and r, the function $\psi_\kappa(t,r)$ is defined by

$$\psi_\kappa(t,r) = (1+bt+r)^{-1}(1+|bt-r|)^{-\kappa+2}.$$

Remark 5.4.1. *The above tools are useful for us to prove global (in time) existence of small data radial solutions to semilinear elastic waves as follows:*

$$\begin{cases} u_{tt} - a^2 \Delta u - \big(b^2 - a^2\big) \nabla \operatorname{div} u = f(u), & x \in \mathbb{R}^3, \ t \in \mathbb{R}_+, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbb{R}^3, \end{cases} \tag{5.36}$$

where $b > a > 0$ and the radial nonlinearities on the right-hand sides can be represented by

$$f(u) = \big(x_1 |x_1^{-1} u^1|^p, x_2 |x_2^{-1} u^2|^p, x_3 |x_3^{-1} u^3|^p\big)^{\mathrm{T}},$$

with $p > 1$. If we take $u(t,x) = x v(t,r)$, then we may immediately transfer (5.36) *to*

$$\begin{cases} v_{tt} - b^2 v_{rr} - \frac{4}{r} b^2 v_r = |v|^p, & r \in \mathbb{R}_+, \ t \in \mathbb{R}_+, \\ (v,v_t)(0,r) = (v_0,v_1)(r), & r \in \mathbb{R}_+. \end{cases}$$

Then, according to [59], global radial solutions exist when $p > p_0(5) = (\sqrt{57}-4)/8$.

5.5. Open problem: Semilinear elastic waves in 3D

To end this chapter, let us discuss the semilinear elastic waves in 3D, namely,

$$\begin{cases} u_{tt} - a^2 \Delta u - \big(b^2 - a^2\big) \nabla \operatorname{div} u = f(u), & x \in \mathbb{R}^3, \ t \in \mathbb{R}_+, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbb{R}^3, \end{cases} \tag{5.37}$$

where the unknown $u=\left(u^1,u^2,u^3\right)^{\mathrm{T}}\in\mathbb{R}^3$ denotes the elastic displacement, t stands for the time and x stands for the space variables. The constants a and b are related to the Lamé constants satisfying $b>a>0$. Precisely, the nonlinear terms can be represented by

$$f(u)=\left(|u^1|^{p_1};|u^2|^{p_2},|u^3|^{p_3}\right)^{\mathrm{T}},$$

with $p_1,p_2,p_3>1$.

In order to understand the Cauchy problem (5.37), we formally take $a=b=1$. In general, the semilinear elastic waves can be reduced to semilinear wave equations with power nonlinearity, i.e.,

$$\begin{cases}u_{tt}-\Delta u=|u|^p, & x\in\mathbb{R}^n,\ t\in\mathbb{R}_+,\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbb{R}^n.\end{cases}\tag{5.38}$$

For $1<p<p_{\text{Kat}}(n)=\frac{n+1}{n-1}$ (here $p_{\text{Kat}}(n)$ denotes the so-called *Kato exponent*) the nonexistence of global in time solutions to (5.38) was proved in [55], providing that initial data is compactly supported. Then, in [52] it was proved that the exponent $p=1+\sqrt{2}$ is the critical exponent for the Cauchy problem (5.38) for $n=3$. In other words, the author of [52] proved a global (in time) existence of classical solutions when $p>1+\sqrt{2}$ and a blow-up of classical solutions when $1<p<1+\sqrt{2}$ under some sign and support condition for initial data. Afterwards, in [100] it was conjectured that the critical exponent for (5.38) is the positive root of the quadratic equation

$$(n-1)p^2-(n+1)p-2=0.$$

Subsequently, in honor of the author of [100] such a conjecture had been named *Strauss' conjecture* and the positive root of the previous quadratic equation in p, denoted by $p_{\text{Str}}(n)$, had be called *Strauss exponent*. In particular, it holds $p_{\text{Str}}(3)=1+\sqrt{2}$. Currently, it is well-known the correctness of Strauss' conjecture. For the proof of the correctness of Strauss' conjecture, we refer to the papers [30, 93, 31] when $n=2$, the papers [52, 93] when $n=3$, and the papers [96, 113, 116, 28, 61] when $n\geqslant 4$.

According to the above discussion about the limit case $a=b=1$, it seems resonable to make the following conjecture.

Conjecture: The critical exponent for the semilinear elastic waves in 3D

$$u_{tt}-a^2\Delta u-\left(b^2-a^2\right)\nabla\operatorname{div}u=\left(|u^1|^{p_1};|u^2|^{p_2},|u^3|^{p_3}\right)^{\mathrm{T}}$$

with $b>a>0$ is the Strauss exponent $p_{\text{Str}}(3)=1+\sqrt{2}$. In other words, global (in time) Sobolev solutions uniquely exist when $p_j>p_{\text{Str}}(3)$, and on the contrary, every local (in time) Sobolev solution blows up in finite time when $1<p_j\leqslant p_{\text{Str}}(3)$ for $j=1,2,3$.

6. Other research results for coupled systems

In this chapter we explain other research results for linear and semilinear coupled systems, which have been studied in the Ph.D. period of Mr. Wenhui Chen. Due to the length of the thesis, we only show the main results of each research topic. The wide discussions of each topic and the proof of each result have been shown in the corresponding paper. One can find them in the following sections.

The chapter is organized as follows:

- In Section 6.1 we show results for global (in time) existence of small data Sobolev solutions and blow-up of Sobolev solutions to weakly coupled systems of semilinear wave equations with distinct scale-invariant terms in the linear part. The results are published in [16].
- In Section 6.2 we show results about several qualitative properties of Sobolev solutions to linear thermoelastic plate equations with friction or structural damping. The results are published in [9].
- In Section 6.3 we present the study of the Cauchy problem for doubly dissipative elastic waves in two space dimensions, where two different damping terms consist of distinct friction or structural damping. The results are published in [8].

6.1. Weakly coupled systems of semilinear wave equations with distinct scale-invariant terms in the linear part

In the paper [16] we consider the following weakly coupled systems of semilinear wave equations with scale-invariant damping and mass terms with different multiplicative constants in the lower order terms:

$$\begin{cases} u_{tt}-\Delta u+\frac{\mu_1}{1+t}u_t+\frac{\nu_1^2}{(1+t)^2}u=|v|^p, & x\in\mathbb{R}^n,\ t\in\mathbb{R}_+,\\ v_{tt}-\Delta v+\frac{\mu_2}{1+t}v_t+\frac{\nu_2^2}{(1+t)^2}v=|u|^q, & x\in\mathbb{R}^n,\ t\in\mathbb{R}_+,\\ (u,u_t,v,v_t)(0,x)=(u_0,u_1,v_0,v_1)(x), & x\in\mathbb{R}^n, \end{cases} \tag{6.1}$$

where $\mu_1,\mu_2,\nu_1^2,\nu_2^2$ are nonnegative constants and $p,q>1$.

As in the case of a single semilinear wave equation with scale-invariant damping and mass term (c.f. [80, 81]), it turns out that the quantities

$$\delta_j:=(\mu_j-1)^2-4\nu_j^2 \quad \text{for} \quad j=1,2,$$

are useful to describe some of the properties of Sobolev solutions of the model (6.1) as, for example, the critical exponents p and q. Let us introduce the notations

$$\alpha_j:=\frac{1}{2}\big(\mu_j+1-\sqrt{\delta_j}\,\big) \quad \text{for} \quad j=1,2.$$

By using the so-called *test function method*, we obtain the following blow-up result. Let us underline that, due to the presence of generally different coefficients in the linear terms of lower order, a new phenomenon appears, that cannot be observed for single equations or for weakly coupled systems with the same linear part (for example, in the case of (6.1) when $\mu_1=\mu_2$ and $\nu_1^2=\nu_2^2$). More precisely, a restriction from below either for p or for q is necessary to get the desired result.

Theorem 6.1.1. *Let $\mu_1, \mu_2, \nu_1^2, \nu_2^2$ be nonnegative constants such that $\delta_1, \delta_2 \geqslant 0$ and let $(u_0, u_1, v_0, v_1) \in \big(H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\big)^2$ be initial data such that*

$$\liminf_{R\to\infty} \int_{B_R} \Big(\frac{1}{2}\big(\mu_1 - 1 + \sqrt{\delta_1}\,\big)u_0(x) + u_1(x)\Big)\mathrm{d}x > 0,$$
$$\liminf_{R\to\infty} \int_{B_R} \Big(\frac{1}{2}\big(\mu_2 - 1 + \sqrt{\delta_2}\,\big)v_0(x) + v_1(x)\Big)\mathrm{d}x > 0.$$

If $p, q > 1$ satisfy the relations

$$\max\Big\{\frac{p+1}{pq-1} - \frac{\alpha_1}{2}; \frac{q+1}{pq-1} - \frac{\alpha_2}{2}\Big\} - \frac{n-1}{2} \geqslant 0,$$
$$\textit{either} \quad p > \frac{1+\alpha_1}{1+\alpha_2} \quad \textit{or} \quad q > \frac{1+\alpha_2}{1+\alpha_1},$$

then, the Cauchy problem (6.1) *has no globally in time weak solution, that is, if*

$$(u, v) \in L^q_{\mathrm{loc}}\big([0, T) \times \mathbb{R}^n\big) \times L^p_{\mathrm{loc}}\big([0, T) \times \mathbb{R}^n\big)$$

is a local in time weak solution with maximal lifespan $T > 0$, then, $T < \infty$.

Now we want to introduce a result about the sufficiency part. We introduce the following notations:

$$\widetilde{p}\,(n, \alpha_1, \alpha_2) := \frac{n + \alpha_1 + 1}{n + \alpha_2 - 1} = 1 + \frac{2 + (\alpha_1 - \alpha_2)}{n + \alpha_2 - 1},$$
$$\widetilde{q}\,(n, \alpha_1, \alpha_2) := \frac{n + \alpha_2 + 1}{n + \alpha_1 - 1} = 1 + \frac{2 + (\alpha_2 - \alpha_1)}{n + \alpha_1 - 1}.$$

In order to prove the global in time existence for small data solutions provided that (p, q) satisfies

$$\max\Big\{\frac{p+1}{pq-1} - \frac{n + \alpha_1 - 1}{2}; \frac{q+1}{pq-1} - \frac{n + \alpha_2 - 1}{2}\Big\} < 0,$$

we may consider separately the following three subcases:

$$p > \widetilde{p}\,(n, \alpha_1, \alpha_2) \ \text{ and } \ q > \widetilde{q}\,(n, \alpha_1, \alpha_2), \tag{6.2}$$
$$p \leqslant \widetilde{p}\,(n, \alpha_1, \alpha_2) \ \text{ and } \ q > \widetilde{q}\,(n, \alpha_1, \alpha_2), \tag{6.3}$$
$$p > \widetilde{p}\,(n, \alpha_1, \alpha_2) \ \text{ and } \ q \leqslant \widetilde{q}\,(n, \alpha_1, \alpha_2). \tag{6.4}$$

More precisely, in the case (6.2) no loss of decay with respect to Sobolev solutions to the corresponding linear Cauchy problem with non-vanishing right-hand side will appear in the decay estimates. On the other hand, in the case, where $p, q > 1$ fulfill (6.3) (respectively (6.4)), because different power source nonlinearities have a different influence on conditions for the global (in time) existence of Sobolev solutions, we allow the effect of loss of decay.

We choose initial data from the classical energy space with additional L^1 regularity, so that the space for the Cauchy data is

$$\mathcal{A}(\mathbb{R}^n) := \big(H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\big) \times \big(L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\big),$$

carrying its corresponding norm

$$\|(f, g)\|_{\mathcal{A}(\mathbb{R}^n)} = \|f\|_{H^1(\mathbb{R}^n)} + \|f\|_{L^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^1(\mathbb{R}^n)}.$$

Theorem 6.1.2. *Let $\mu_1, \mu_2 > 1$, ν_1^2, ν_2^2 be nonnegative constants such that $\delta_1, \delta_2 > (n+1)^2$. Let us assume $p, q > 1$, satisfying $2 \leqslant p, q$ and $p, q \leqslant \frac{n}{n-2}$ if $n \geqslant 3$, such that*

$$p > \widetilde{p}\,(n, \alpha_1, \alpha_2) \text{ and } q > \widetilde{q}\,(n, \alpha_1, \alpha_2).$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for any $(u_0, u_1, v_0, v_1) \in \mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)$ with

$$\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)} \leqslant \varepsilon_0$$

there is a uniquely determined energy solution

$$(u, v) \in \big(\mathcal{C}\big([0, \infty), H^1(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0, \infty), L^2(\mathbb{R}^n)\big)\big)^2$$

to the Cauchy problem (6.1). *Furthermore, the solution (u, v) satisfies the following decay estimates:*

$$\begin{aligned}
\|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^n)} + \|u_t(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_1}\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)},\\
\|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_1+1}\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)},\\
\|\nabla v(t, \cdot)\|_{L^2(\mathbb{R}^n)} + \|v_t(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_2}\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)},\\
\|v(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_2+1}\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)}.
\end{aligned}$$

Theorem 6.1.3. *Let $\mu_1, \mu_2 > 1$, ν_1^2, ν_2^2 be nonnegative constants such that $\delta_1, \delta_2 > (n+1)^2$. Let us assume $p, q > 1$, satisfying $2 \leqslant p, q$ and $p, q \leqslant \frac{n}{n-2}$ if $n \geqslant 3$, such that*

$$\begin{aligned}
&p \leqslant \widetilde{p}\,(n, \alpha_1, \alpha_2) \text{ and } q > \widetilde{q}\,(n, \alpha_1, \alpha_2),\\
&\frac{q+1}{pq-1} - \frac{n+\alpha_2-1}{2} < 0.
\end{aligned}$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for any $(u_0, u_1, v_0, v_1) \in \mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)$ with

$$\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)} \leqslant \varepsilon_0$$

there is a uniquely determined energy solution

$$(u, v) \in \big(\mathcal{C}\big([0, \infty), H^1(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0, \infty), L^2(\mathbb{R}^n)\big)\big)^2$$

to the Cauchy problem (6.1). *Furthermore, the solution (u, v) satisfies the following estimates:*

$$\begin{aligned}
\|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^n)} + \|u_t(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_1+\gamma}\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)},\\
\|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_1+1+\gamma}\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)},\\
\|\nabla v(t, \cdot)\|_{L^2(\mathbb{R}^n)} + \|v_t(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_2}\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)},\\
\|v(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_2+1}\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)},
\end{aligned}$$

where

$$0 < \gamma = \gamma(p, n, \alpha_1, \alpha_2) := \begin{cases} (n+\alpha_2-1)(\widetilde{p}\,(n, \alpha_1, \alpha_2) - p) & \text{if } p < \widetilde{p}\,(n, \alpha_1, \alpha_2),\\ \epsilon & \text{if } p = \widetilde{p}\,(n, \alpha_1, \alpha_2), \end{cases}$$

represents the loss of decay in comparison with the corresponding decay estimates for the solution u to the linear Cauchy problem with vanishing right hand side, $\epsilon > 0$ being an arbitrarily small constant in the limit case $p = \widetilde{p}\,(n, \alpha_1, \alpha_2)$.

Theorem 6.1.4. *Let $\mu_1, \mu_2 > 1$, ν_1^2, ν_2^2 be nonnegative constants such that $\delta_1, \delta_2 > (n+1)^2$. Let us assume $p, q > 1$, satisfying $2 \leqslant p, q$ and $p, q \leqslant \frac{n}{n-2}$ if $n \geqslant 3$, such that*

$$\begin{aligned} &p > \widetilde{p}(n, \alpha_1, \alpha_2) \ \text{ and } \ q \leqslant \widetilde{q}(n, \alpha_1, \alpha_2), \\ &\frac{p+1}{pq-1} - \frac{n+\alpha_1-1}{2} < 0. \end{aligned}$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for any $(u_0, u_1, v_0, v_1) \in \mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n)$ with

$$\|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n)\times\mathcal{A}(\mathbb{R}^n)} \leqslant \varepsilon_0$$

there is a uniquely determined energy solution

$$(u, v) \in \big(\mathcal{C}\big([0,\infty), H^1(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,\infty), L^2(\mathbb{R}^n)\big)\big)^2$$

to (6.1)*. Furthermore, the solution (u, v) satisfies the following estimates:*

$$\begin{aligned} \|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_1} \|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n)\times\mathcal{A}(\mathbb{R}^n)}, \\ \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_1+1} \|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n)\times\mathcal{A}(\mathbb{R}^n)}, \\ \|\nabla v(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \|v_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_2+\bar{\gamma}} \|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n)\times\mathcal{A}(\mathbb{R}^n)}, \\ \|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim (1+t)^{-\frac{n}{2}-\alpha_2+1+\bar{\gamma}} \|(u_0, u_1, v_0, v_1)\|_{\mathcal{A}(\mathbb{R}^n)\times\mathcal{A}(\mathbb{R}^n)}, \end{aligned}$$

where

$$0 < \bar{\gamma} = \bar{\gamma}(q, n, \alpha_1, \alpha_2) := \begin{cases} (n+\alpha_1-1)(\widetilde{q}(n,\alpha_1,\alpha_2) - q) & \text{if } q < \widetilde{q}(n,\alpha_1,\alpha_2), \\ \epsilon & \text{if } q = \widetilde{q}(n,\alpha_1,\alpha_2), \end{cases}$$

represents the loss of decay in comparison with the corresponding decay estimates for the solution v to the linear Cauchy problem with vanishing right hand side, $\epsilon > 0$ being an arbitrarily small constant in the limit case $q = \widetilde{q}(n, \alpha_1, \alpha_2)$.

Finally, we give a conclusion. Combining the results from Theorems 6.1.1, 6.1.2, 6.1.3 and 6.1.4 we have that

$$\max\Big\{\frac{p+1}{pq-1} - \frac{\alpha_1}{2}; \frac{q+1}{pq-1} - \frac{\alpha_2}{2}\Big\} = \frac{n-1}{2}$$

is a relation for the critical exponents p and q for the weakly coupled system (6.1) provided that the coefficients satisfy $\delta_1, \delta_2 > (n+1)^2$ in the sense we explained. Actually, one can slightly improve this result up to the range $\delta_1, \delta_2 \geqslant (n+1)^2$ modulo a (possible) further arbitrarily small loss of decay rate with respect to the case $\delta_1, \delta_2 > (n+1)^2$ in Theorems 6.1.3 and 6.1.4

In the case $0 < \delta_1 < (n+1)^2$ or $0 < \delta_2 < (n+1)^2$ we are not able to obtain a sharp result as in the above mentioned case by using $L^2 - L^2$ estimates with additional L^1 regularity and working in classical energy spaces, due to the fact that the first-order derivatives have a weaker decay rate (c.f. Theorems 4.6 and 4.7 in [80] for further details).

6.2. Linear thermoelastic plate equations with different damping mechanisms

In the paper [9] we are concerned with the following Cauchy problem for thermoelastic plate equations in $\mathbb{R}^n$, $n \geqslant 1$, where the heat conduction is modeled by Fourier's law:

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta\theta + (-\Delta)^\sigma u_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ \theta_t - \Delta\theta - \Delta u_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t, \theta)(0, x) = (u_0, u_1, \theta_0)(x), & x \in \mathbb{R}^n, \end{cases} \tag{6.5}$$

where $\sigma \in [0,2]$. To be more specific, $\sigma = 0$ stands for the system with *friction or external damping*, $\sigma \in (0,2]$ stands for the system with *structural damping*, especially, $\sigma = 2$ stands for the system with *Kelvin-Voigt type damping*.

By introducing the quantities

$$\begin{aligned} U(t,x) &:= \left(u_t + |D|^2 u, u_t - |D|^2 u, \theta\right)^{\mathrm{T}}(t,x), \\ U_0(x) &:= \left(u_1 + |D|^2 u_0, u_1 - |D|^2 u_0, \theta_0\right)^{\mathrm{T}}(x), \\ P_{U_0} &:= \int_{\mathbb{R}^n} U_0(x)\mathrm{d}x, \end{aligned}$$

we may reduce (6.5) to the following evolution system

$$\begin{cases} U_t - A_0 \Delta U + A_1 (-\Delta)^{\sigma} U = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ U(0,x) = U_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{6.6}$$

where the coefficient matrices are given by

$$A_0 = \frac{1}{2}\begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & -2 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad A_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we derive representations of solutions in the Fourier space by applying WKB analysis and a suitable diagonalization procedure.

Now, we collect the following results about qualitative properties of Sobolev solutions:

- Gevrey smoothing of solutions if $\sigma \in [0,2)$ (see Table 6.1);
- L^2 well-posedness for the Cauchy problem (6.5) such that

$$U \in \left(\mathcal{C}\left([0,\infty), L^2(\mathbb{R}^n)\right)\right)^3 \quad \text{if we assume} \quad U_0 \in \left(L^2(\mathbb{R}^n)\right)^3;$$

- energy estimates of solutions

$$\|U(t,\cdot)\|_{(\dot{H}^s(\mathbb{R}^n))^3} \lesssim (1+t)^{-\frac{(2-m)n+2ms}{2mK}} \|U_0\|_{(H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n))^3},$$

  where $s \geqslant 0$, $m \in [1,2]$, and

$$\|U(t,\cdot)\|_{(\dot{H}^s(\mathbb{R}^n))^3} \lesssim (1+t)^{-\frac{n+2(s+\delta)}{2K}} \|U_0\|_{(H^s(\mathbb{R}^n)\cap L^{1,\delta}(\mathbb{R}^n))^3} + (1+t)^{-\frac{n+2s}{2K}} |P_{U_0}|,$$

  where $s \geqslant 0$, $\delta \in (0,1]$. Here some numbers K appears (specified in the table below);
- $L^p - L^q$ estimates not necessary on the conjugate line of the form

$$\|U(t,\cdot)\|_{(\dot{H}^s_q(\mathbb{R}^n))^3} \lesssim (1+t)^{-\frac{s}{K}-\frac{n}{K}\left(\frac{1}{p}-\frac{1}{q}\right)} \|U_0\|,$$

  for suitable p, q and some number K (specified in the table below). Here $\|U_0\|$ corresponds to initial data measured in an appropriate norm, which is based on the L^p norm;
- diffusion phenomena with data belonging to different function spaces;
- asymptotic profiles of solutions in a framework of $L^{1,1}$ data such that

$$t^{-\frac{n+2s}{2K}} |P_{U_0}| \lesssim \|U(t,\cdot)\|_{(\dot{H}^s(\mathbb{R}^n))^3} \lesssim t^{-\frac{n+2s}{2K}} \|U_0\|_{(H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n))^3}$$

  for $t \gg 1$, where $|P_{U_0}| \neq 0$, $s \geqslant 0$ and some numbers K are chosen in Table 6.1.

<table>
<tr><th></th><th>$\sigma=0$</th><th>$\sigma\in(0,1)$</th><th>$\sigma=1$</th><th>$\sigma\in(1,3/2]$</th><th>$\sigma\in(3/2,2)$</th><th>$\sigma=2$</th></tr>
<tr><td>Gevrey smoothing</td><td colspan="4">$\left(\Gamma^1\left(\mathbb{R}^n\right)\right)^3$ (analytic smoothing)</td><td>$\left(\Gamma^{1/(4-2\sigma)}\left(\mathbb{R}^n\right)\right)^3$</td><td>-</td></tr>
<tr><td>Energy estimates</td><td colspan="2">$K=4-2\sigma$</td><td colspan="4">$K=2$</td></tr>
<tr><td>L^p-L^q estimates</td><td colspan="2">$K=4-2\sigma$ and $1\leqslant p\leqslant 2\leqslant q\leqslant\infty$</td><td>$K=2$ and $1\leqslant p\leqslant q\leqslant\infty$</td><td colspan="3">$K=2$ and $1\leqslant p\leqslant 2\leqslant q\leqslant\infty$</td></tr>
<tr><td>Diffusion phenomena (dif. phe.)</td><td>double dif. phe.</td><td>triple dif. phe.</td><td>-</td><td colspan="3">single dif. phe.</td></tr>
<tr><td>Asymptotic profiles</td><td colspan="2">$K=4-2\sigma$</td><td colspan="4">$K=2$</td></tr>
</table>

Tab. 6.1.: Summary of qualitative properties of Sobolev solutions

We should point out that when $K=4$, friction has a dominant influence in the corresponding decay estimates; when $K=4-2\sigma$, structural damping has a dominant influence in the corresponding decay estimates; when $K=2$, thermal dissipation generated by Fourier's law has a dominant influence in the corresponding decay estimates.

Finally, we found that this method, in general, to derive sharp asymptotic profiles in this paper can be probably applied to the Cauchy problem for other coupled systems in elastic material, for instance, dissipative elastic waves, thermoelastic systems and thermodiffusion systems.

6.3. Doubly dissipative elastic waves in 2D

Let us consider the following Cauchy problem for doubly dissipative elastic waves in two space dimensions:

$$\begin{cases} u_{tt}-a^2\Delta u-\left(b^2-a^2\right)\nabla \operatorname{div} u+(-\Delta)^{\rho}u_t+(-\Delta)^{\theta}u_t=0, & x\in\mathbb{R}^2,\ t\in\mathbb{R}_+,\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbb{R}^2, \end{cases} \tag{6.7}$$

where the unknown $u=u(t,x)\in\mathbb{R}^2$ denotes the elastic displacement. The positive constants a and b in (6.7) are related to the Lamé constants and fulfill $b>a>0$. Moreover, the parameters ρ and θ in (6.7) satisfy $0\leqslant\rho<1/2<\theta\leqslant 1$. This problem is strongly related to the open problem proposed in [44]. In the paper [8] we give an answer in the two-dimensional case.

Applying a change of variables, we may transfer (6.7) to the following evolution system:

$$\begin{cases} U_t+\frac{1}{2}B_0(-\Delta)^{\rho}U+iB_1(-\Delta)^{1/2}U+\frac{1}{2}B_0(-\Delta)^{\theta}U=0, & x\in\mathbb{R}^2,\ t\in\mathbb{R}_+,\\ U(0,x)=U_0(x), & x\in\mathbb{R}^2, \end{cases} \tag{6.8}$$

where the coefficient matrices B_0 and B_1 are respectively given by

$$B_0=\begin{pmatrix} 1&0&1&0\\ 0&1&0&1\\ 1&0&1&0\\ 0&1&0&1 \end{pmatrix}\quad\text{and}\quad B_1=\begin{pmatrix} -b&0&0&0\\ 0&-a&0&0\\ 0&0&b&0\\ 0&0&0&a \end{pmatrix}.$$

Then, by applying WKB analysis associated to a multi-step diagonalization method, we may derive energy estimates.

Theorem 6.3.1. *Let us consider the Cauchy problem* (6.8) *with* $0 \leqslant \rho < 1/2 < \theta \leqslant 1$ *and* $U_0 \in \left(H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\right)^4$, *where* $s \geqslant 0$ *and* $m \in [1,2]$. *Then, the following decay estimates hold:*

$$\|U(t,\cdot)\|_{(\dot{H}^s(\mathbb{R}^2))^4} \lesssim (1+t)^{-\frac{s}{2-2\rho}-\frac{2-m}{m(2-2\rho)}} \|U_0\|_{(H^s(\mathbb{R}^2)\cap L^m(\mathbb{R}^2))^4}.$$

Remark 6.3.1. *We remark that the energy estimates for doubly dissipative elastic waves* (6.7) *in Theorem 6.3.1 are the same as for damped elastic waves with damping term* $(-\Delta)^\rho u_t$ *for* $\rho \in [0,1/2)$ *in Theorems 7.2 and 7.3 in [88]. In other words, the decay rate is only determined by the damping term* $(-\Delta)^\rho u_t$ *with* $\rho \in [0,1/2)$ *in* (6.7).

Concerning diffusion phenomena, we first introduce the corresponding reference systems in the cases $\rho+\theta<1$, $\rho+\theta=1$ and $\rho+\theta>1$, respectively. Firstly, we introduce the matrices

$$M_1 := \begin{pmatrix} b^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & -b^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_1 := \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

which will be used later. We define different reference systems between the following three cases.

- In the case $\rho+\theta<1$, we define $\widetilde{U}=\widetilde{U}(t,x;\rho,\theta)$ as the solution to

$$\begin{cases} \widetilde{U}_t + M_1(-\Delta)^{1-\rho}\widetilde{U} + M_2(-\Delta)^\rho\widetilde{U} + M_2(-\Delta)^\theta\widetilde{U} = 0, & x\in\mathbb{R}^2,\ t\in\mathbb{R}_+, \\ \widetilde{U}(0,x) = T_1^{-1}U_0(x), & x\in\mathbb{R}^2. \end{cases} \tag{6.9}$$

- In the case $\rho+\theta=1$, we define $\widetilde{U}=\widetilde{U}(t,x;\rho,\theta)$ as the solution to

$$\begin{cases} \widetilde{U}_t + (M_1+M_2)(-\Delta)^{1-\rho}\widetilde{U} + M_2(-\Delta)^\rho\widetilde{U} = 0, & x\in\mathbb{R}^2,\ t\in\mathbb{R}_+, \\ \widetilde{U}(0,x) = T_1^{-1}U_0(x), & x\in\mathbb{R}^2. \end{cases} \tag{6.10}$$

- In the case $\rho+\theta>1$, we define $\widetilde{U}=\widetilde{U}(t,x;\rho,\theta)$ as the solution to

$$\begin{cases} \widetilde{U}_t + M_1(-\Delta)^{1-\rho}\widetilde{U} + M_2(-\Delta)^\rho\widetilde{U} = 0, & x\in\mathbb{R}^2,\ t\in\mathbb{R}_+, \\ \widetilde{U}(0,x) = T_1^{-1}U_0(x), & x\in\mathbb{R}^2. \end{cases} \tag{6.11}$$

Let us now provide some explanations for these reference systems.

In the case when $\rho+\theta<1$, we find that the reference system (6.9) is made up of three different operators, i.e., $(-\Delta)^{1-\rho}$, $(-\Delta)^\rho$ and $(-\Delta)^\theta$. In this case, the damping terms $(-\Delta)^\rho u_t$ and $(-\Delta)^\theta u_t$ with $0\leqslant\rho<1/2<\theta\leqslant1$ in (6.7) have an influence on the diffusion structure at the same time. But this effect does not appear in the other case $\rho+\theta\geqslant1$.

However, we find that when $\rho+\theta\geqslant1$, the reference system (6.9) is changed into (6.10) and (6.11). Obviously, these reference systems are only made up of two different operators $(-\Delta)^\rho$ and $(-\Delta)^{1-\rho}$, whose structures are similar as those in the reference system for elastic waves with damping term $(-\Delta)^\rho u_t$ for $\rho\in[0,1/2)$. In other words, the diffusion structure is dominant by the damping term $(-\Delta)^\rho u_t$ with $0\leqslant\rho<1/2$.

According to the above discussions, we observe a new threshold of diffusion structure for doubly dissipative elastic waves, that is $\rho+\theta=1$. In other words, the structure of the reference system will be changed if the parameters change from $\rho+\theta<1$ to $\rho+\theta\geqslant1$.

More precisely, the next theorem holds.

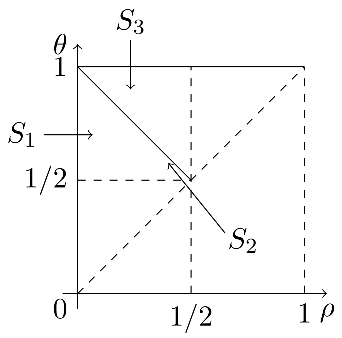
Theorem 6.3.2. *Let us consider the Cauchy problem* (6.8) *with* $0 \leqslant \rho < 1/2 < \theta \leqslant 1$ *and* $U_0 \in L^m(\mathbb{R}^n)$ *with* $m \in [1,2]$. *Then, the following refined estimates hold:*

$$\left\|\chi_{\mathrm{int}}(D)\left(U(t,\cdot)-T_1\widetilde{U}(t,\cdot;\rho,\theta)\right)\right\|_{(\dot{H}^s(\mathbb{R}^2))^4} \lesssim (1+t)^{-\frac{s}{2-2\rho}-\frac{2-m}{m(2-2\rho)}-\Theta(\rho,\theta)}\|U_0\|_{(L^m(\mathbb{R}^2))^4},$$

where the function $\Theta = \Theta(\rho,\theta)$ *is defined by*

$$\Theta(\rho,\theta) := \begin{cases} \frac{2\theta-1}{2-2\rho} & \text{if } (\rho,\theta) \in S_1, \\ \frac{1-2\rho}{2-2\rho} & \text{if } (\rho,\theta) \in S_2 \cup S_3. \end{cases}$$

Here the regions S_1, S_2, S_3 *are given by Figure 6.1 as follows:*



$$S := \{(\rho,\theta) : 0 \leqslant \rho < 1/2 < \theta \leqslant 1\}$$
$$S_1 := \{(\rho,\theta) \in S : \rho+\theta < 1\}$$
$$S_2 := \{(\rho,\theta) \in S : \rho+\theta = 1\}$$
$$S_3 := \{(\rho,\theta) \in S : \rho+\theta > 1\}$$

Fig. 6.1.: Range of a pair of exponent (ρ,θ)

To end this section, we mention that we are also able to derive optimal decay rates in estimates of solutions in the $\dot{H}^s$ norm with $s \geqslant 0$ in a suitable framework of weighted L^1 spaces.

7. Concluding remarks and further research topics

In this chapter further studies during the Ph.D. period of Mr. Wenhui Chen, will be collected. Due to the length of the thesis, we just present the main results of each topic. The remaining part of this chapter is organized as follows:

- In Section 7.1 we present decay properties and asymptotic profiles for Sobolev solutions to generalized thermoelastic plate equations with Fourier's law of heat conduction. The results are introduced in [64].
- In Section 7.2 we present blow-up results of energy solutions to semilinear Moore-Gibson-Thompson equations with power nonlinearity or nonlinearity of derivative type. The results are introduced in [15, 14].
- In Section 7.3 we present results about global existence of small data Sobolev solutions and blow-up of Sobolev solution to semilinear strongly damped wave equations with mixed nonlinearity in exterior domains. The results are introduced in [12, 11].
- In Section 7.4 we present blow-up results of energy solutions to semilinear wave equations with a nonlinear memory term. The results are introduced in [13].

7.1. Generalized thermoelastic plate equations

In recent years, Cauchy problems for evolution-parabolic coupled systems have caught a lot of attention. Particularly, the so-called $\alpha-\beta$ system describes few physical models, including second-order thermoelastic equations, thermoelastic plate equations and linear viscoelastic equations.

Let us consider $\alpha-\beta$ system with $\alpha=\beta$. More precisely, we study the following Cauchy problem for generalized thermoelastic plate equations:

$$\begin{cases} u_{tt}+\mathcal{A}u-\mathcal{A}^{\alpha}v=0, & x\in\mathbb{R}^n,\ t\in\mathbb{R}_+,\\ v_t+\mathcal{A}^{\alpha}v+\mathcal{A}^{\alpha}u_t=0, & x\in\mathbb{R}^n,\ t\in\mathbb{R}_+,\\ (u,u_t,v)(0,x)=(u_0,u_1,v_0)(x), & x\in\mathbb{R}^n, \end{cases} \tag{7.1}$$

where $\alpha\in[0,1]$, and $\mathcal{A}=(-\Delta)^{\sigma}$ with $\sigma\in[1,\infty)$.

Let us introduce the quantity $w=w(t,x)$ such that

$$w:=\left(u_t+i(-\Delta)^{\sigma/2}u,u_t-i(-\Delta)^{\sigma/2}u,v\right)^{\mathrm{T}}, \tag{7.2}$$

which is the solution to the next evolution system:

$$\begin{cases} w_t-B_0(-\Delta)^{\sigma/2}w-B_1(-\Delta)^{\sigma\alpha}w=0, & x\in\mathbb{R}^n,\ t\in\mathbb{R}_+,\\ w(0,x)=w_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{7.3}$$

where the coefficient matrices are defined by

$$B_0=\begin{pmatrix} i & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & 0\end{pmatrix} \quad\text{and}\quad B_1=\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 1\\ -\frac{1}{2} & -\frac{1}{2} & -1\end{pmatrix}.$$

Employing WKB analysis, we may explore the following sharp pointwise estimate in the Fourier space, which is useful for us to derive energy estimates in Theorem 7.1.1.

Proposition 7.1.1. *The solution $\hat{w}=\hat{w}(t,\xi)$, which is the partial Fourier transform of $w=w(t,x)$, satisfies the following pointwise estimates for any $\xi\in\mathbb{R}^n$ and $t\geqslant 0$:*

$$|\hat{w}(t,\xi)|\lesssim \mathrm{e}^{-c\rho(|\xi|)t}|\hat{w}_0(\xi)|,$$

where the key function $\rho=\rho(|\xi|)$ is defined by

$$\rho(|\xi|):=\begin{cases}\frac{|\xi|^{2\sigma-2\sigma\alpha}}{(1+|\xi|^2)^{2\sigma-4\sigma\alpha}} & \text{if } \alpha\in[0,1/2],\\ \frac{|\xi|^{6\sigma\alpha-2\sigma}}{(1+|\xi|^2)^{4\sigma\alpha-2\sigma}} & \text{if } \alpha\in(1/2,1],\end{cases}$$

with a positive constant $c>0$.

Remark 7.1.1. *Comparing with the pointwise estimates in Theorem 1.2 in [107], fortunately, we take advantage of the quantity* (7.2) *and a multi-step diagonalization procedure, which allow us to avoid the constraint condition $\partial_{x_j}\phi^k-\partial_{x_k}\phi^j=0$ for initial data.*

Let us now give some explanations for decay properties, which are introduced by the pointwise estimates in Proposition 7.1.1.

- We observe that the threshold for decay properties for $|\xi|\to 0$ is $\alpha=1/2$. Precisely, in the case when $\alpha\in[0,1/2]$, the key function fulfills $\rho(|\xi|)\approx|\xi|^{2\sigma-2\sigma\alpha}$ for $|\xi|\to 0$, and in the case when $\alpha\in(1/2,1]$, the key function satisfies $\rho(|\xi|)\approx|\xi|^{6\sigma\alpha-2\sigma}$ for $|\xi|\to 0$. Thus, we expect that there exist different decay estimates between these two cases.
- The threshold for decay properties for $|\xi|\to\infty$ is $\alpha=1/3$. In other words, we see $\rho(|\xi|)\approx|\xi|^{-2\sigma(1-3\alpha)}$ in the case when $\alpha\in[0,1/3)$, which leads to decay properties of regularity-loss type. However, for the other case when $\alpha\in[1/3,1]$, we expect that the regularity-loss structure is removed.
- All in all, the threshold for decay properties can be described by the numbers

$$\alpha=\frac{1}{2}\text{ for }|\xi|\to 0\quad\text{and}\quad\alpha=\frac{1}{3}\text{ for }|\xi|\to\infty.$$

Theorem 7.1.1. *Suppose that initial data $w_0\in\left(H^s(\mathbb{R}^n)\cap L^{1,\kappa}(\mathbb{R}^n)\right)^3$ with $s\geqslant 0$ and $\kappa\in[0,1]$. Then, the Sobolev solutions to* (7.3) *with $\sigma\geqslant 1$ satisfy the following estimates:*

$$\|w(t,\cdot)\|_{(\dot{H}^{s_0}(\mathbb{R}^n))^3}\lesssim\begin{cases}(1+t)^{-\frac{n+2(s_0+\kappa)}{2(2\sigma-2\sigma\alpha)}}\|w_0\|_{(L^{1,\kappa}(\mathbb{R}^n))^3}+(1+t)^{-\frac{\ell}{2\sigma(1-3\alpha)}}\|w_0\|_{(H^{s_0+\ell}(\mathbb{R}^n))^3}\\ \qquad+(1+t)^{-\frac{n+2s_0}{2(2\sigma-2\sigma\alpha)}}|P_{w_0}| & \text{if } \alpha\in[0,1/3),\\ (1+t)^{-\frac{n+2(s_0+\kappa)}{2(2\sigma-2\sigma\alpha)}}\|w_0\|_{(L^{1,\kappa}(\mathbb{R}^n))^3}+\mathrm{e}^{-ct}\|w_0\|_{(H^{s_0}(\mathbb{R}^n))^3}\\ \qquad+(1+t)^{-\frac{n+2s_0}{2(2\sigma-2\sigma\alpha)}}|P_{w_0}| & \text{if } \alpha\in[1/3,1/2],\\ (1+t)^{-\frac{n+2(s_0+\kappa)}{2(6\sigma\alpha-2\sigma)}}\|w_0\|_{(L^{1,\kappa}(\mathbb{R}^n))^3}+\mathrm{e}^{-ct}\|w_0\|_{(H^{s_0}(\mathbb{R}^n))^3}\\ \qquad+(1+t)^{-\frac{n+2s_0}{2(6\sigma\alpha-2\sigma)}}|P_{w_0}| & \text{if } \alpha\in(1/2,1],\end{cases}$$

where $0\leqslant s_0+\ell\leqslant s$ with $s_0\geqslant 0$ and $\ell\geqslant 0$. Here $c>0$ is a positive constant and we denote $P_f:=\int_{\mathbb{R}^n}f(x)\,\mathrm{d}x$.

To end this section, we summarize the results in [64] to show the asymptotic profiles of solutions. Due to the length of the section, we show the improvements (i.e., the consideration of $\|w(t,\cdot)-\widetilde{w}(t,\cdot)\|_{(\dot{H}^s(\mathbb{R}^n))^3}$, where $\widetilde{w}=\widetilde{w}(t,x)$ is the solution to the corresponding reference system) of decay rate and regularity of initial data only. Please see Figure 7.1.

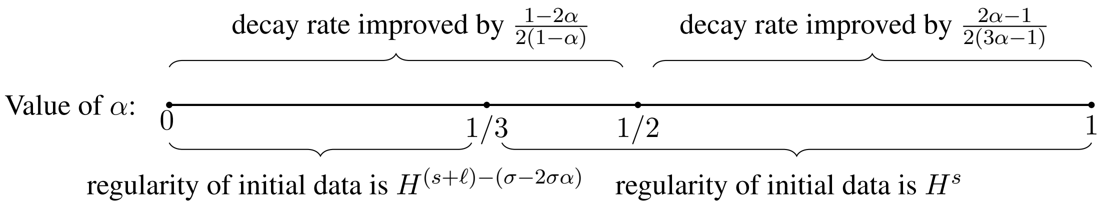


Fig. 7.1.: Decay rates and regularity of initial data

Next, we turn to generalized thermoelastic plate equations with additional structural damping, namely, to

$$\begin{cases} u_{tt} + \mathcal{A}u - \mathcal{A}^\alpha v + \mathcal{A}^{1/2}u_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ v_t + \mathcal{A}^\alpha v + \mathcal{A}^\alpha u_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t, v)(0, x) = (u_0, u_1, v_0)(x), & x \in \mathbb{R}^n, \end{cases} \tag{7.4}$$

and analyze the influence of structural damping on qualitative properties of solutions. We find that the regularity-loss structure is destroyed by structural damping. Moreover, we are able to give some applications of our results on thermoelastic plate equations and the Moore-Gibson-Thompson equations with friction.

7.2. Semilinear Moore-Gibson-Thompson equations

Over the last years, the Moore-Gibson-Thompson (MGT) equation, a linearization of a model for the wave propagation in viscous thermally relaxing fluid has been widely studied. This model is realized through the third-order hyperbolic equation

$$\tau u_{ttt} + u_{tt} - c^2\Delta u - b\Delta u_t = 0. \tag{7.5}$$

In the physical context of acoustic waves, the unknown function $u = u(t, x)$ denotes the scalar acoustic velocity, c denotes the speed of sound and τ denotes the thermal relaxation. Besides, the coefficient $b = \beta c^2$ is related to the diffusivity of the sound with $\beta \in (0, \tau]$. According to the previous studies by using the theory of semigroups, we may refer to the limit case $\beta = \tau$ as to the conservative case.

In this project (c.f. [15, 14]), we consider the semilinear Cauchy problem for MGT equations in the conservative case, namely,

$$\begin{cases} \beta u_{ttt} + u_{tt} - \Delta u - \beta\Delta u_t = f(u, u_t; p), & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ (u, u_t, u_{tt})(0, x) = \varepsilon(u_0, u_1, u_2)(x), & x \in \mathbb{R}^n, \end{cases} \tag{7.6}$$

with a positive constant β, where the nonlinearity on the right-hand side can be chosen as

$$f(u, u_t; p) = \begin{cases} |u|^p, & \text{i.e., power nonlinearity,} \\ |u_t|^p, & \text{i.e., power nonlinearity of derivative type,} \end{cases}$$

with $p > 1$. For the sake of simplicity, we normalized the speed of the sound by putting $c^2 = 1$. Moreover, $\varepsilon > 0$ is a parameter describing the smallness of initial data.

Our aim in this research project is to prove blow-up of energy solutions to (7.6) with some condition on initial data and exponent p. We propose a slicing procedure of the domain of integration by taking inspiration from [2], even though the sequence of the parameters, that characterizes the slicing of the domain of integration, has a completely different structure. Our result is the first attempt to include an unbounded exponential multiplier in an iteration argument for proving blow-up results for hyperbolic semilinear models.

Before introducing the blow-up results, we mention that the local (in time) existence of solutions is given by the following proposition.

Proposition 7.2.1. *Let $n \geqslant 1$. Let us consider $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ compactly supported with $\operatorname{supp} u_j \subset B_R$ for any $j = 0, 1, 2$ and for some $R > 0$. We assume $p > 1$ such that $p \leqslant n/(n-2)$ when $n \geqslant 3$. Then, there exists a positive T and a uniquely determined local (in time) mild solution*

$$u \in \mathcal{C}([0,T], H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0,T], H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0,T], L^2(\mathbb{R}^n))$$

to (7.6) *satisfying* $\operatorname{supp} u(t, \cdot) \subset B_{R+t}$ *for any* $t \in [0,T]$.

7.2.1. Blow-up of energy solutions for MGT with power nonlinearity

The aim of [15] is to prove blow-up of local (in time) energy solutions to (7.6) with $|u|^p$ in the sub-Strauss case, that is for $1 < p < p_{\mathrm{Str}}(n)$, under suitable conditions for the Cauchy data, and to derive an upper bound estimate for the lifespan. To do this, we introduce first the definition of energy solutions to (7.6) with $|u|^p$. Here the so-called *Strauss exponent* $p_{\mathrm{Str}}(n)$ is the critical exponent of semilinear wave equation with $|u|^p$, where $p_{\mathrm{Str}}(n)$ is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Definition 7.2.1. *Let $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. We say that u is an energy solution to* (7.6) *with $|u|^p$ on $[0,T)$ if*

$$u \in \mathcal{C}([0,T), H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0,T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0,T), L^2(\mathbb{R}^n)) \cap L^p_{\mathrm{loc}}([0,T) \times \mathbb{R}^n)$$

satisfy $u(0, \cdot) \in H^2(\mathbb{R}^n)$ and the integral identity

$$\begin{aligned}
&\beta \int_{\mathbb{R}^n} u_{tt}(t,x)\psi(t,x)\,\mathrm{d}x + \int_{\mathbb{R}^n} u_t(t,x)\psi(t,x)\,\mathrm{d}x \\
&- \beta\varepsilon \int_{\mathbb{R}^n} u_2(x)\psi(0,x)\,\mathrm{d}x - \varepsilon \int_{\mathbb{R}^n} u_1(x)\psi(0,x)\,\mathrm{d}x \\
&+ \beta \int_0^t \int_{\mathbb{R}^n} \big(\nabla u_t(s,x) \cdot \nabla\psi(s,x) - u_{tt}(s,x)\psi_t(s,x)\big)\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}^n} \big(\nabla u(s,x) \cdot \nabla\psi(s,x) - u_t(s,x)\psi_t(s,x)\big)\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}^n} |u(s,x)|^p \psi(s,x)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{7.7}$$

holds for any $\psi \in \mathcal{C}_0^\infty([0,T) \times \mathbb{R}^n)$ and any $t \in [0,T)$.

Theorem 7.2.1. *Let us consider $p > 1$ such that*

$$\begin{cases} p < \infty & \text{if } n = 1, \\ p < p_{\mathrm{Str}}(n) & \text{if } n \geqslant 2. \end{cases}$$

Let $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ be nonnegative and compactly supported functions with supports contained in B_R for some $R > 0$ such that u_0 is not identically zero. Let

$$u \in \mathcal{C}([0,T), H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0,T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0,T), L^2(\mathbb{R}^n)) \cap L^p_{\mathrm{loc}}([0,T) \times \mathbb{R}^n)$$

be an energy solution on $[0,T)$ to the Cauchy problem (7.6) *with $|u|^p$ according to Definition 7.2.1 with lifespan $T = T(\varepsilon)$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, u_2, n, p, R, \beta)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan*

$$T(\varepsilon) \leqslant C\varepsilon^{-\frac{2p(p-1)}{\theta(p,n)}}$$

holds, where the constant C is an independent of ε and

$$\theta(p,n) := 2 + (n+1)p - (n-1)p^2. \tag{7.8}$$

Then, to show the result for the critical case, we define energy solution.

Definition 7.2.2. *Let* $(u_0,u_1,u_2)\in H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. *We say that*

$$u\in\mathcal{C}\big([0,T),H^2(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,T),H^1(\mathbb{R}^n)\big)\cap\mathcal{C}^2\big([0,T),L^2(\mathbb{R}^n)\big)\cap L^p_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)$$

is an energy solution on $[0,T)$ *if* u *fulfills* $u(0,\cdot)\in H^2(\mathbb{R}^n)$ *and the integral relation*

$$\begin{aligned}&\int_{\mathbb{R}^n}u_t(t,x)\,\psi(t,x)\,\mathrm{d}x-\varepsilon\int_{\mathbb{R}^n}u_1(x)\,\psi(0,x)\,\mathrm{d}x\\&+\int_0^t\int_{\mathbb{R}^n}\big(\nabla_x u(s,x)\cdot\nabla_x\psi(s,x)-u_t(s,x)\,\psi_s(s,x)\big)\,\mathrm{d}x\,\mathrm{d}s\\&=\varepsilon\int_0^t\mathrm{e}^{-s/\beta}\int_{\mathbb{R}^n}\psi(s,x)\,(u_2(x)-\Delta u_0(x))\,\mathrm{d}x\,\mathrm{d}s\\&\quad+\frac{1}{\beta}\int_0^t\int_0^s\mathrm{e}^{(\tau-s)/\beta}\int_{\mathbb{R}^n}\psi(s,x)|u(\tau,x)|^p\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s\end{aligned}\tag{7.9}$$

for any $\psi\in\mathcal{C}_0^\infty([0,T)\times\mathbb{R}^n)$ *and any* $t\in[0,T)$.

Theorem 7.2.2. *Let* $n\geqslant 2$ *and* $p=p_{\mathrm{Str}}(n)$. *Let us assume that* $(u_0,u_1,u_2)\in H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ *are nonnegative and compactly supported functions with supports contained in* B_R *for some* $R>0$ *such that* u_0 *is not identically zero and* $u_2-\Delta u_0$ *is nonnegative. Let*

$$u\in\mathcal{C}\big([0,T),H^2(\mathbb{R}^n)\big)\cap\mathcal{C}^1\big([0,T),H^1(\mathbb{R}^n)\big)\cap\mathcal{C}^2\big([0,T),L^2(\mathbb{R}^n)\big)\cap L^p_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)$$

be an energy solution on $[0,T)$ *according to Definition 7.2.2 with lifespan* $T=T(\varepsilon)$. *Then, there exists a positive constant* $\varepsilon_0=\varepsilon_0(u_0,u_1,n,p,R,\beta)$ *such that for any* $\varepsilon\in(0,\varepsilon_0]$ *the energy solution* u *blows up in finite time. Moreover, the upper bound estimate for the lifespan*

$$T(\varepsilon)\leqslant\exp\Big(C\varepsilon^{-p(p-1)}\Big)$$

holds, where the constant $C>0$ *is independent of* ε.

7.2.2. Blow-up of energy solutions for MGT with nonlinearity of derivative type

The main result of [14] consists of a blow-up result for (7.6) with $|u_t|^p$ when the power of the nonlinear term is in the sub-Glassey range (including the case $p=p_{\mathrm{Gla}}(n)$). Here the so-called *Glassey exponent* $p_{\mathrm{Gla}}(n):=(n+1)/(n-1)$ is the critical exponent of the classical semilinear wave equation with nonlinearity $|u_t|^p$ on the right-hand side.

Before stating the main result of this paper, let us introduce a suitable notion of energy solutions to the Cauchy problem (7.6).

Definition 7.2.3. *Let* $(u_0,u_1,u_2)\in H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. *We say that* u *is an energy solution of* (7.6) *with right-hand side* $|u_t|^p$ *if*

$$\begin{aligned}&u\in\mathcal{C}([0,T),H^2(\mathbb{R}^n))\cap\mathcal{C}^1([0,T),H^1(\mathbb{R}^n))\cap\mathcal{C}^2([0,T),L^2(\mathbb{R}^n))\\&\textit{such that}\quad u_t\in L^p_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)\end{aligned}$$

satisfies $u(0,\cdot)\in H^2(\mathbb{R}^n)$ *and the integral relation*

$$\begin{aligned}
&\beta\int_{\mathbb{R}^n}u_{tt}(t,x)\phi(t,x)\,\mathrm{d}x+\int_{\mathbb{R}^n}u_t(t,x)\phi(t,x)\,\mathrm{d}x\\
&-\beta\varepsilon\int_{\mathbb{R}^n}u_2(x)\phi(0,x)\,\mathrm{d}x-\varepsilon\int_{\mathbb{R}^n}u_1(x)\phi(0,x)\,\mathrm{d}x\\
&+\beta\int_0^t\int_{\mathbb{R}^n}(\nabla u_t(s,x)\cdot\nabla\phi(s,x)-u_{tt}(s,x)\phi_t(s,x))\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{R}^n}(\nabla u(s,x)\cdot\nabla\phi(s,x)-u_t(s,x)\phi_t(s,x))\,\mathrm{d}x\,\mathrm{d}s\\
&=\int_0^t\int_{\mathbb{R}^n}|u_t(s,x)|^p\phi(s,x)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}\tag{7.10}$$

holds for any $\phi\in\mathcal{C}_0^\infty([0,T)\times\mathbb{R}^n)$ *and any* $t\in(0,T)$.

Theorem 7.2.3. *Let us consider* $p>1$ *such that*

$$\begin{cases}p<\infty & \text{if } n=1,\\ p\leqslant p_{\mathrm{Gla}}(n) & \text{if } n\geqslant 2.\end{cases}$$

Let $(u_0,u_1,u_2)\in H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ *be nonnegative and compactly supported functions with supports contained in* B_R *for some* $R>0$ *such that* u_1 *or* u_2 *is not identically zero.*
Let u *be the energy solution to the Cauchy problem* (7.6) *with* $|u_t|^p$ *and lifespan* $T(\varepsilon)$*. Then, there exists a positive constant* $\varepsilon_0=\varepsilon_0(u_0,u_1,u_2,n,p,R,\beta)$ *such that for any* $\varepsilon\in(0,\varepsilon_0]$ *the solution* u *blows up in finite time. Furthermore, the upper bound estimate for the lifespan*

$$T(\varepsilon)\leqslant\begin{cases}C\varepsilon^{-(\frac{1}{p-1}-\frac{n-1}{2})^{-1}} & \text{if } 1<p<p_{\mathrm{Gla}}(n),\\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } p=p_{\mathrm{Gla}}(n),\end{cases}$$

holds, where the constant $C>0$ *is independent of* ε.

7.3. Semilinear strongly damped wave equations in exterior domains in 2D

In the recent papers [12, 11] we consider the following initial boundary value problem for two-dimensional semilinear strongly damped wave equations with mixed nonlinearity in an exterior domain:

$$\begin{cases}u_{tt}-\Delta u-\Delta u_t=|u|^p+|u_t|^q, & x\in\Omega,\ t\in\mathbb{R}_+,\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\ t\in\mathbb{R}_+,\end{cases}\tag{7.11}$$

with $p,q>1$, where Ω is an exterior domain. Our aim in this research project is to understand the global (in time) existence of small data energy solutions and blow-up of weak solutions. Therefore, let us divide the discussion into two parts.

7.3.1. Global existence for higher-order energy solutions

Let us assume that $\Omega\subset\mathbb{R}^2$ is a smooth exterior domain. For simplicity we assume $0\notin\overline{\Omega}$.

The main approach in proving global existence for higher-order energy solutions is the use of weighted energy method. Let us point out that the study of the exterior problem with $|u|^p+|u_t|^q$ is not simply a generalization of what happens for the exterior problem with $|u|^p$. On the one hand, the application of the Gagliardo-Nirenberg inequality with a weighted function (e.g. Lemma 2.3 in [42]) allows us to estimate the nonlinear term $|u_t|^q$ in a weighted L^q space by its gradient in L^2 space and weighted L^2 space. Thus, we need to construct a new weighted energy for nonlinear exterior problems. On the other hand, in order to estimate a new energy in exponentially weighted spaces, we have to construct the weighted function with a suitable parameter.

Motivated by [42], we introduce the weight function as follows:

$$\psi(t,x) := \frac{1}{\rho(1+t)^{\rho}} + \frac{|x|^2}{2(1+t)^{2+\rho}}$$

with a constant $\rho > \rho_0$, where $\rho_0 > 0$ is defined by the positive root of the quadratic equation $2\rho_0^2 + 3\rho_0 - 8 = 0$. By direct computations, the next properties are fulfilled:

$$\psi_t(t,x) < 0, \quad \Delta\psi(t,x) = \frac{2}{(1+t)^{2+\rho}} \quad \text{and} \quad -\psi_t(t,x) \leqslant \frac{C_\rho}{1+t}\psi(t,x),$$

where the positive constant C_ρ is independent of x and t. Furthermore, it holds that

$$|\nabla\psi(t,x)|^2 - \psi_t(t,x)|\nabla\psi(t,x)|^2 - |\psi_t(t,x)|^2 \leqslant 0.$$

We assume that $\varepsilon > 0$ is an auxiliary constant satisfying

$$\frac{4\rho+14}{(2+\rho)(2\rho+3)} \leqslant \varepsilon < 1,$$

for all $\rho > \rho_0 \approx 1.386$. Next, we define the time-dependent function

$$d(x) := |x| \log(B|x|)$$

with a positive constant B such that $\inf\limits_{x\in\Omega} |x| \geqslant 2/B > 0$.

Moreover, we introduce a norm for initial data, namely

$$\begin{aligned} \mathcal{J}[u_0,u_1] := \sum_{j=0,1} \Big(\|u_j\|^2_{L^2(\Omega)} + \|\nabla u_j\|^2_{L^2(\Omega)} \Big) + \|\Delta u_0\|^2_{L^2(\Omega)} \\ + \|d(\cdot)\Delta u_0\|^2_{L^2(\Omega)} + \|d(\cdot)u_1\|^2_{L^2(\Omega)} + I_w[u_0,u_1], \end{aligned}$$

where

$$I_w[u_0,u_1] := \int_\Omega \mathrm{e}^{2\psi(0,x)} \left(|\nabla u_1(x)|^2 + |\Delta u_0(x)|^2 + |u_1(x)|^2 + |\nabla u_0(x)|^2 \right) \mathrm{d}x.$$

Theorem 7.3.1. *Let us assume $p, q > 6 + 2\rho_0$. Then, there exists a constant $\varepsilon_0 > 0$ such that for any*

$$(u_0,u_1) \in \big(H^2(\Omega) \cap H^1_0(\Omega)\big) \times \big(H^2(\Omega) \cap H^1_0(\Omega)\big)$$

with $\mathcal{J}[u_0,u_1] \leqslant \varepsilon_0$, there is a uniquely determined higher-order energy solution

$$u \in \mathcal{C}\big([0,\infty), H^2_0(\Omega)\big) \cap \mathcal{C}^1\big([0,\infty), H^1(\Omega)\big)$$

to (7.11)*. Furthermore, the solution satisfies the following estimates:*

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\Omega)} &\leqslant C\mathcal{J}[u_0,u_1], \\ \|\mathcal{D}u(t,\cdot)\|^2_{L^2(\Omega)} &\leqslant C(1+t)^{-1}\mathcal{J}[u_0,u_1], \\ \left\| \mathrm{e}^{\psi(t,\cdot)}\mathcal{D}u(t,\cdot) \right\|^2_{L^2(\Omega)} &\leqslant C\mathcal{J}[u_0,u_1], \end{aligned}$$

where the space-time differential operator is denoted by $\mathcal{D} := (\partial_t, \nabla, \nabla\partial_t, \Delta)$.

7.3.2. Blow-up of weak solutions in arbitrary dimensions

Let us choose $\Omega \subset \mathbb{R}^n$ as an exterior domain whose obstacle $\mathcal{O} \subset \mathbb{R}^n$ with $n \geqslant 1$ and smooth compact boundary $\partial\Omega$. Without loss of generality, we assume that $0 \in \mathcal{O} \subset\subset B(R)$, where $B(R) := \{x \in \mathbb{R}^n : |x| < R\}$ denotes a ball with radius R centered at the origin.

Before showing our main results. We should underline the definition of a weak solution.

Proposition 7.3.1. *Let $T > 0$, $(u_0, u_1) \in L^1_{\text{loc}}(\Omega) \times L^1_{\text{loc}}(\Omega)$ and the right-hand side of (7.11) belongs to $L^1((0,T), L^1_{\text{loc}}(\Omega))$. A function u is said to be a weak solution to (7.11) if it satisfies the relation*

$$\begin{aligned}&\int_0^T \int_\Omega F(t,x)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_1(x)\varphi(0,x)\,\mathrm{d}x - \int_\Omega u_0(x)\Delta\varphi(0,x)\,\mathrm{d}x - \int_\Omega u_0(x)\varphi_t(0,x)\,\mathrm{d}x\\ &= \int_0^T \int_\Omega u(0,x)\varphi_{tt}(0,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega u(t,x)\Delta\varphi_t(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega u(t,x)\Delta\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all compactly supported test function $\varphi \in \mathcal{C}^2([0,T] \times \Omega)$ such that $\varphi(T,x) = 0$ and $\varphi_t(T,x) = 0$.

To prove blow-up of weak solutions, we apply the test function method with a suitable weight function, which will be introduced later. Let us give some harmonic functions in each dimension. The existence of these functions has been introduced in [117].

Lemma 7.3.1. *There exists a function $\phi_0 = \phi_0(x) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ for $n \geqslant 3$ satisfying the boundary value problem*

$$\begin{cases} \Delta\phi_0(x) = 0, & x \in \Omega,\\ \phi_0(x) = 0, & x \in \partial\Omega,\\ \phi_0(x) \to 1, & |x| \to \infty. \end{cases}$$

Moreover, the function ϕ_0 satisfies $0 < \phi_0(x) < 1$ for all $x \in \Omega$, and $\phi_0(x) \geqslant C$ for all $|x| \gg 1$. Furthermore, for all $|x| \gg 1$ we have $|\nabla\phi_0(x)| \leqslant C|x|^{1-n}$.

Lemma 7.3.2. *There exists a function $\phi_0 = \phi_0(x) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ for $n = 2$ satisfying the boundary value problem*

$$\begin{cases} \Delta\phi_0(x) = 0, & x \in \Omega,\\ \phi_0(x) = 0, & x \in \partial\Omega,\\ \phi_0(x) \to +\infty, & |x| \to \infty \text{ and } \phi_0(x) \text{ increases at the rate of } \ln(|x|). \end{cases}$$

Moreover, the function ϕ_0 satisfies $0 < \phi_0(x) \leqslant C\ln(|x|)$ for all $x \in \Omega$, and $\phi_0(x) \geqslant C$ for all $|x| \gg 1$. Furthermore, for all $|x| \gg 1$, $|\nabla\phi_0(x)| \leqslant C|x|^{-1}$.

Lemma 7.3.3. *There exists a function $\phi_0(x) = Cx$ for $x \geqslant 0$, which solves the boundary value problem*

$$\begin{cases} \Delta\phi_0(x) = 0, & x > 0,\\ \phi_0(x) = 0, & x = 0,\\ \phi_0(x) \to +\infty, & |x| \to \infty \text{ and } \phi_0(x) \text{ increases at the rate of linear function } x. \end{cases}$$

Moreover, the function ϕ_0 satisfies that there exist two positive constants C_1 and C_2 such that, for all $x > 0$, we have $C_1 x \leqslant \phi_0(x) \leqslant C_2 x$.

Theorem 7.3.2. *Let us assume $(u_0, u_1) \in \big(H^1_0(\Omega) \cap L^1(\Omega)\big) \times H^1(\Omega)$ and $\phi_0 u_1 \in L^1(\Omega)$ such that*

$$\int_\Omega \phi_0(x) u_1(x)\,\mathrm{d}x > 0,$$

where ϕ_0 is defined in Lemmas 7.3.1, 7.3.2 and 7.3.3.
The case when $n=1$: If one of the following conditions is fulfilled:

$$\begin{cases} 1<p\leqslant 1+\alpha_1\approx 2.28 \quad \textit{and} \quad 1<q, \\ 1<p\leqslant \frac{2\alpha+1}{2\alpha-1} \quad \textit{and} \quad 1<q\leqslant \frac{\alpha+1}{2} \quad \textit{for any}\, \alpha<\alpha_1, \\ 1<p\leqslant \alpha+1 \quad \textit{and} \quad 1<q\leqslant \frac{2\alpha+1}{2\alpha} \quad \textit{for any}\, \alpha_1<\alpha<\alpha_2, \\ 1<p \quad \textit{and} \quad 1<q\leqslant \frac{1+\alpha_2}{2}\approx 1.3, \end{cases}$$

where $\alpha_1=(1+\sqrt{17})/4$ is the positive root of $2\alpha^2-\alpha-2=0$, and $\alpha_2=(1+\sqrt{5})/2$ is the positive root of $\alpha^2-\alpha-1=0$, then weak solutions to (7.11) *according to Definition 7.3.1 blow up in finite time.*

The case when $n\geqslant 2$: If one of the following condition is fulfilled:

$$\begin{cases} 1<p<3 \quad \textit{or} \quad 1<q<1+\frac{1}{2} & \textit{for}\, n=2, \\ 1<p\leqslant 1+\frac{2}{n-1} \quad \textit{or} \quad 1<q\leqslant 1+\frac{1}{n} & \textit{for}\, n\geqslant 3, \end{cases}$$

then weak solutions to (7.11) *defined in Definition 7.3.1 blow up in finite time.*

7.4. Semilinear wave equations with a nonlinear memory term

In the recent paper [13], we investigate the blow-up dynamic for local in time energy solutions to the semilinear wave equation with a nonlinearity of memory type, namely,

$$\begin{cases} u_{tt}-\Delta u=\frac{1}{\Gamma(1-\gamma)}\int_0^t (t-s)^{-\gamma}|u(s,x)|^p\,\mathrm{d}s, & x\in\mathbb{R}^n,\ t\in(0,T), \\ (u,u_t)(0,x)=\varepsilon(u_0,u_1)(x), & x\in\mathbb{R}^n, \end{cases} \tag{7.12}$$

where $p>1$, $\gamma\in(0,1)$, Γ denotes the Euler integral of second kind and ε is a parameter describing the size of initial data.

Let us introduce the following quadratic equation:

$$\frac{n-1}{2}p^2-\left(\frac{n+1}{2}+1-\gamma\right)p-1=0, \tag{7.13}$$

where $\gamma\in(0,1)$ and $p>1$. For any $n\geqslant 2$ and any $\gamma\in(0,1)$ we denote by $p=p_0(n,\gamma)$ the positive root of (7.13), that is,

$$p_0(n,\gamma):=\frac{n+3-2\gamma+\sqrt{n^2+(14-4\gamma)n+(3-2\gamma)^2-8}}{2(n-1)}.$$

For $n=1$ we set formally $p_0(1,\gamma)=\infty$ for any $\gamma\in(0,1)$.

If $p_{\mathrm{Str}}(n)$ denotes the Strauss exponent, that is the critical exponent for the classical semilinear wave equation with power nonlinearity, whose analytic expression can be derived from the quadratic equation

$$\frac{n-1}{2}p^2-\frac{n+1}{2}p-1=0$$

(also in this case, one may formally put $p_{\mathrm{Str}}(1)=\infty$), then, we notice that

$$\lim_{\gamma\to 1^-}p_0(n,\gamma)=p_{\mathrm{Str}}(n).$$

Before introducing a result on blow-up of energy solutions, we introduce the notion of energy solutions to the Cauchy problem (7.12).

Definition 7.4.1. *Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. We say that*

$$u \in \mathcal{C}\left([0,T), H^1(\mathbb{R}^n)\right) \cap \mathcal{C}^1\left([0,T), L^2(\mathbb{R}^n)\right)$$

is an energy solution of (7.12) *on $[0,T)$, if u fulfills $u(0,\cdot) \in H^1(\mathbb{R}^n)$ and the integral relation*

$$\begin{aligned}
&\int_{\mathbb{R}^n} u_t(t,x)\psi(t,x)\,\mathrm{d}x - \varepsilon \int_{\mathbb{R}^n} u_1(x)\psi(0,x)\,\mathrm{d}x \\
&+ \int_0^t \int_{\mathbb{R}^n} \left(\nabla u(s,x)\cdot\nabla\psi(s,x) - u_t(s,x)\psi_t(s,x)\right)\mathrm{d}x\,\mathrm{d}s \\
&= \frac{1}{\Gamma(1-\gamma)} \int_0^t \int_{\mathbb{R}^n} \psi(s,x) \int_0^s (s-\tau)^{-\gamma}|u(\tau,x)|^p\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{7.14}$$

for any $\psi \in \mathcal{C}_0^\infty\left([0,T)\times\mathbb{R}^n\right)$ and any $t \in [0,T)$.

Theorem 7.4.1. *Let us consider $p > 1$ such that*

$$\begin{cases} p < \infty & \text{if } n = 1, \\ p < p_0(n,\gamma) & \text{if } n \geqslant 2. \end{cases}$$

Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ be nonnegative and compactly supported functions with supports contained in B_R for some $R > 0$ such that $u_0 \not\equiv 0$. Let

$$\begin{aligned}
&u \in \mathcal{C}\left([0,T), H^1(\mathbb{R}^n)\right) \cap \mathcal{C}^1\left([0,T), L^2(\mathbb{R}^n)\right) \\
&\text{such that } \int_0^t (t-s)^{-\gamma}|u(s,x)|^p\,\mathrm{d}s \in L^1_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)
\end{aligned}$$

be an energy solution on $[0,T)$ to (7.12) *according to Definition 7.4.1 with lifespan $T = T(\varepsilon)$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0,u_1,n,p,\gamma,R)$ such that for any $\varepsilon \in (0,\varepsilon_0]$ the energy solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan*

$$T(\varepsilon) \leqslant C\varepsilon^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}}$$

holds, where the positive constant C is independent of ε and

$$\Upsilon(p,n,\gamma) := 2 + (n+1+2(1-\gamma))p - (n-1)p^2. \tag{7.15}$$

Theorem 7.4.2. *Let $n \geqslant 2$ and $p = p_0(n,\gamma)$. Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ be nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$. Let*

$$\begin{aligned}
&u \in \mathcal{C}\big([0,T), H^1(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,T), L^2(\mathbb{R}^n)\big) \\
&\text{such that } \int_0^t (t-s)^{-\gamma}|u(s,x)|^p\,\mathrm{d}s \in L^1_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)
\end{aligned}$$

be an energy solution on $[0,T)$ to (7.12) *according to Definition 7.4.1 with lifespan $T = T(\varepsilon)$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0,u_1,n,p,\gamma,R)$ such that for any $\varepsilon \in (0,\varepsilon_0]$ the energy solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan*

$$T(\varepsilon) \leqslant \exp\left(C\varepsilon^{-p(p-1)}\right)$$

holds, where C is an independent of ε, positive constant.

A.Notions-Guide to the reader

A.1. General notation

We use C and c as arbitrary constants throughout the thesis. It may differ at each occurrence, unless explicitly stated otherwise.

The natural numbers $\mathbb{N} = \{1, 2, \dots\}$ do not contain zero, hence, we introduce additionally the set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We use $\mathbb{R}_+$ to denote all non-negative real numbers and $\mathbb{R}^n$ is the set of all vectors $x = (x_1, \dots, x_n)^{\mathrm{T}}$ with $x_j \in \mathbb{R}$, $j = 1, \dots, n$. Furthermore, Re z and Im z are the real part and the imaginary part of $z \in \mathbb{C}$, respectively.

The symbol D is used to denote $D = -i\nabla$, $\mathrm{div} = \nabla^{\mathrm{T}} = (\partial_{x_1}, \dots, \partial_{x_n})$, and $D_t = -i\partial_t$, $i = \sqrt{-1}$. We write $\partial_t u = u_t$ for a function $u = u(t, \cdot)$ and $\Delta := \Delta_x$ denotes the Laplace operator with respect to $x \in \mathbb{R}^n$. Moreover, $|D|^\sigma$ stands for the pseudo-differential operator with symbol $|\xi|^\sigma$, and $\langle D\rangle^\sigma$ stands for the pseudo-differential operator with symbol $\langle\xi\rangle^\sigma$. Here $\langle\xi\rangle = \sqrt{|\xi|^2 + 1}$.

Bracket symbols with special meaning are

$\langle\cdot\rangle$	denotes Japanese bracket $\langle\cdot\rangle = \sqrt{|\cdot|^2 + 1}$,
$\lvert \cdot \rvert$	denotes the absolute value of a scalar expression, for a vector the Euclidean and for a matrix the Frobenius norm,
$\lceil\cdot\rceil$	denotes the smallest integer large than a given number, that is, $\lceil x\rceil = \min\{m \in \mathbb{Z} : x \leqslant m\}$,
$[\cdot,\cdot]$	denotes the commutator for two matrices.

We use the following relations: let f, g be nonnegative functions satisfying $f \leqslant C_1 g$ and $f \geqslant C_2 g$ for all arguments, where C_1, C_2 are positive constants, then we use the notation $f \lesssim g$ and $f \gtrsim g$, respectively.

For matrices we use the notations:

$I_{k\times k}$	denotes the identity matrix of dimensions $k \times k$,
$\mathbf{0}_{k\times k}$	denotes a zero matrix of dimensions $k \times k$,
$\mathrm{diag}(A_1, \dots, A_k)$	denotes a diagonal matrix with the scalars $A_1, \dots, A_k$ on the diagonal.

We denote the symbol $\oplus$ between Jordan matrices $J_{l_j}(\lambda_j)$ as follows:

$$J_{l_1}(\lambda_1) \oplus J_{l_2}(\lambda_2) \oplus \dots \oplus J_{l_n}(\lambda_n) := \begin{pmatrix} J_{l_1}(\lambda_1) & & & \\ & J_{l_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{l_n}(\lambda_n) \end{pmatrix},$$

where the Jordan matrix is defined by

$$J_{l_j}(\lambda_j) := \begin{pmatrix} \lambda_j & 1 & & & \\ & \lambda_j & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_j & 1 \\ & & & & \lambda_j \end{pmatrix}.$$

The Landau symbol O will as usual be used for describing an asymptotic upper bound of a function in terms of author, simpler function, that is, we say $f(x) = O(g(x))$ as $x \to x_0$ for function $f, g : X \subset \mathbb{R}^n \to \mathbb{C}, x_0 \in X'$ being from the derived set of X, if there exists a $\delta > 0$ such that

$$|f(x)| \lesssim |g(x)| \quad \text{for } x \in X \text{ with } |x - x_0| < \delta.$$

Analogous definitions are employed for $|x| \to \infty$ and $x \to \pm\infty$ if $n = 1$. Moreover, we will use the O notation for matrix-valued functions $f = f(x)$ as well and this means that each component is of the order of $g = g(x)$.

The Fourier transform for functions $f \in \mathcal{S}(\mathbb{R}^n)$ or $f \in L^1(\mathbb{R}^n)$ is defined as

$$\mathcal{F}(f(x)) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} f(x)\mathrm{d}x,$$
$$\mathcal{F}^{-1}(f(\xi)) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} f(\xi)\mathrm{d}\xi.$$

For more general f, such as $f \in L^2(\mathbb{R}^n)$, the corresponding natural definitions are employed. Moreover, we often use $\mathcal{F}_{x\to\xi}(f(t,x))$ and $\mathcal{F}_{\xi\to x}(f(t,\xi))$ to denote the partial Fourier transform and its inverse transform.

A.2. Function spaces

We have collected function spaces frequently used in the thesis together with a short definition:

$L^p(\mathbb{R}^n)$	Lebesgue space, $p \geqslant 1$,
$W^s_p(\mathbb{R}^n)$	Sobolev-Slobodeckij space,
$H^s_p(\mathbb{R}^n)$	Bessel potential space, $H^s_p(\mathbb{R}^n) = \langle D\rangle^{-s} L^p(\mathbb{R}^n)$,
$\dot{H}^s_p(\mathbb{R}^n)$	Riesz potential space, $\dot{H}^s_p(\mathbb{R}^n) = \lvert D\rvert^{-s} L^p(\mathbb{R}^n) \subseteq \mathcal{S}'_{\mathcal{P}}(\mathbb{R}^n)$,
$H^s(\mathbb{R}^n)$	special case for $p = 2$: $H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$,
$\dot{H}^s(\mathbb{R}^n)$	special case for $p = 2$: $\dot{H}^s(\mathbb{R}^n) = \dot{H}^s_2(\mathbb{R}^n)$,
$\mathcal{C}^k(\mathbb{R}^n)$	space of k-times continuously differentiable functions,
$\mathcal{C}^\infty(\mathbb{R}^n)$	projective limit $\mathcal{C}^\infty(\mathbb{R}^n) = \cap_{k=0}^\infty \mathcal{C}^k(\mathbb{R}^n)$,
$\mathcal{C}^\infty_0(\mathbb{R}^n)$	space of $\mathcal{C}^\infty(\mathbb{R}^n)$-functions with compact support,
$\Gamma^\kappa(\mathbb{R}^n)$	Gevrey space $\Gamma^\kappa = \{f(x) \in L^2(\mathbb{R}^n) : \exp(\gamma\langle\xi\rangle^{1/\kappa})\mathcal{F}(f)(\xi) \in L^2(\mathbb{R}^n)\}$,
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space of rapidly decreasing functions,
$\mathcal{S}'(\mathbb{R}^n)$	space of tempered distributions,
$\mathcal{S}'_{\mathcal{P}}(\mathbb{R}^n)$	tempered distributions modulo polynomials,
$\mathcal{Z}(\mathbb{R}^n)$	space of Schwartz functions with all moments vanishing,
$\mathcal{Z}'(\mathbb{R}^n)$	topological dual of $\mathcal{Z}(\mathbb{R}^n)$ which can be canonically identified with the factor space $\mathcal{S}'(\mathbb{R}^n)\backslash\mathcal{P}$,
$\mathcal{M}^q_p(\mathbb{R}^n)$	space of multiplier with parameter $p, q \in \mathbb{R}$,

B. Basic tools

B.1. Interpolation theory

Proposition B.1.1. *(Riesz-Thorin Interpolation Theorem) Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$. If T is a linear continuous operator in the space $\mathcal{L}\big(L^{p_0}(\mathbb{R}^n) \to L^{q_0}(\mathbb{R}^n)\big) \cap \big(L^{p_1}(\mathbb{R}^n) \to L^{q_1}(\mathbb{R}^n)\big)$, then*

$$T \in \mathcal{L}\big(L^{p_\theta}(\mathbb{R}^n) \to L^{q_\theta}(\mathbb{R}^n)\big)$$

for any $\theta \in (0,1)$, where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Moreover, the following norm estimate holds:

$$\|T\|_{\mathcal{L}(L^{p_\theta}(\mathbb{R}^n) \to L^{q_\theta}(\mathbb{R}^n))} \leqslant \|T\|^{1-\theta}_{\mathcal{L}(L^{p_0}(\mathbb{R}^n) \to L^{q_0}(\mathbb{R}^n))} \|T\|^{\theta}_{\mathcal{L}(L^{p_1}(\mathbb{R}^n) \to L^{q_1}(\mathbb{R}^n))}.$$

One can see the proof in Appendix A in [85].

B.2. Some inequalities in fractional Sobolev spaces

Let us introduce Bessel and Riesz potential spaces first.

Let $s \in \mathbb{R}$ and $1 < p < \infty$. Let us introduce $J_s = (I-\Delta)^{s/2} = \langle D \rangle^s$ and $I_s = (-\Delta)^{s/2} = |D|^s$, respectively. Then,

$$\begin{aligned} H^s_p(\mathbb{R}^n) &= \big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|J_s f\|_{L^p(\mathbb{R}^n)} = \|f\|_{H^s_p(\mathbb{R}^n)} < \infty \big\}, \\ \dot{H}^s_p(\mathbb{R}^n) &= \big\{ f \in \mathcal{Z}'(\mathbb{R}^n) : \|I_s f\|_{L^p(\mathbb{R}^n)} = \|f\|_{\dot{H}^s_p(\mathbb{R}^n)} < \infty \big\}, \end{aligned}$$

are said to be Bessel and Riesz potential spaces, respectively.

Proposition B.2.1. *(Classical Gagliardo-Nirenberg Inequality) Let $j, m \in \mathbb{N}$ with $j < m$, and let $f \in \mathcal{C}^m_0(\mathbb{R}^n)$. Let $\beta = \beta_{j,m} \in [j/m, 1]$ with $p, q, r \in [1, \infty]$ satisfying*

$$j - \frac{n}{q} = \Big(m - \frac{n}{r}\Big)\beta - \frac{n}{p}(1-\beta).$$

Then, it holds

$$\big\|D^j f\big\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|^{1-\beta}_{L^p(\mathbb{R}^n)} \big\|D^m f\big\|^{\beta}_{L^r(\mathbb{R}^n)}$$

provided that $(m - n/r) - j \notin \mathbb{N}$. If $(m - n/r) - j \in \mathbb{N}$, the Gagliardo-Nirenberg inequality holds provided that $\beta_{j,m} \in [j/m, 1)$.

The proof of it can be found in [27], Part I, Theorem 9.3.

Proposition B.2.2. *(Fractional Gagliardo-Nirenberg Inequality) Let $p, p_0, p_1 \in (1,\infty)$ and $\kappa \in [0, s)$ with $s > 0$. Then, it holds for all $f \in L^{p_0}(\mathbb{R}^n) \cap \dot{H}^s_{p_1}(\mathbb{R}^n)$*

$$\|f\|_{\dot{H}^\kappa_p(\mathbb{R}^n)} \lesssim \|f\|^{1-\beta}_{L^{p_0}(\mathbb{R}^n)} \|f\|^{\beta}_{\dot{H}^s_{p_1}(\mathbb{R}^n)},$$

where $\beta = \beta_{\kappa,s} = \big(\frac{1}{p_0} - \frac{1}{p} + \frac{\kappa}{n}\big) \big\backslash \big(\frac{1}{p_0} - \frac{1}{p_1} + \frac{s}{n}\big)$ and $\beta \in [\kappa/s, 1]$.

The proof of this result may be found in [35].

Proposition B.2.3. *(Fractional Leibniz Rule) Let $s > 0$, $1 \leqslant r \leqslant \infty$, $1 < p_1, p_2, q_1, q_2 \leqslant \infty$ satisfying*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then, it holds for $f \in \dot{H}^s_{p_1}(\mathbb{R}^n) \cap L^{q_1}(\mathbb{R}^n)$ and $g \in \dot{H}^s_{q_2}(\mathbb{R}^n) \cap L^{q_2}(\mathbb{R}^n)$

$$\|fg\|_{\dot{H}^s_r(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s_{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)} + \|f\|_{L^{q_1}(\mathbb{R}^n)} \|g\|_{\dot{H}^s_{q_2}(\mathbb{R}^n)}.$$

The proof of above inequality can be found in [32].

Proposition B.2.4. *(Fractional Chain Rule) Let $s > 0$, $p > \lceil s \rceil$, $1 < r, r_1, r_2 < \infty$ satisfying*

$$\frac{1}{r} = \frac{p-1}{r_1} + \frac{1}{r_2}.$$

Then, it holds for $f \in \dot{H}^s_{r_2}(\mathbb{R}^n) \cap L^{r_1}(\mathbb{R}^n)$

$$\big\| \pm f|f|^{p-1} \big\|_{\dot{H}^s_r(\mathbb{R}^n)} + \big\| |f|^p \big\|_{\dot{H}^s_r(\mathbb{R}^n)} \lesssim \|f\|^{p-1}_{L^{r_1}(\mathbb{R}^n)} \|f\|_{\dot{H}^s_{r_2}(\mathbb{R}^n)}.$$

We can find the proof in [81].

Proposition B.2.5. *(Fractional Powers Rule) Let $r \in (1, \infty)$, $p > 1$ and $s \in [0, p)$. Then, it holds for $f \in \dot{H}^s_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*

$$\big\| \pm f|f|^{p-1} \big\|_{\dot{H}^s_r(\mathbb{R}^n)} + \big\| |f|^p \big\|_{\dot{H}^s_r(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s_r(\mathbb{R}^n)} \|f\|^{p-1}_{L^\infty(\mathbb{R}^n)}.$$

Proposition B.2.6. *Let $r \in (1, \infty)$ and $s > 0$. Then, it holds for all $f, g \in \dot{H}^s_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*

$$\|fg\|_{\dot{H}^s_r(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s_r(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{\dot{H}^s_r(\mathbb{R}^n)}.$$

The above two propositions can be found in [81].

Proposition B.2.7. *Let $0 < 2s^* < n < 2s$. Then, for any function $f \in \dot{H}^{s^*}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$ one has*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leqslant \|f\|_{\dot{H}^{s^*}(\mathbb{R}^n)} + \|f\|_{\dot{H}^s(\mathbb{R}^n)}.$$

The proof of this statement was given in [23].

B.3. Estimates for Fourier multipliers

Proposition B.3.1. *Let f be a measurable function. Moreover, we suppose the following relation with suitable positive constants C, $b \in (1, \infty)$ and all positive ℓ:*

$$\text{meas}\big\{\xi \in \mathbb{R}^n : |f(\xi)| \geqslant \ell\big\} \leqslant C\ell^{-b}.$$

Then, $f \in \mathcal{M}^q_p(\mathbb{R}^n)$ if $1 < p \leqslant 2 \leqslant q < \infty$ and $1/p - 1/q = 1/b$.

One can find the proof in [92].

B.4. Littman type lemma

When we derive $L^1 - L^\infty$ estimates for solution of a given linear equation, then the following result is useful. Its proof is based on the method of the stationary phase (see [99, 101]) and can be found for example in [25].

Proposition B.4.1. *(Littman Type Lemma) Let us consider for $\tau > \tau_0$, here $\tau_0 > 0$ is a large number, the oscillating integral*

$$\mathcal{F}^{-1}_{\eta\to x}\big(\mathrm{e}^{-i\tau|\eta|}f(\eta)\big).$$

We assume the amplitude function $f \in \mathcal{C}_0^\infty\big(\mathbb{R}^n\big)$ with support in $\{\eta \in \mathbb{R}^n : |\eta| \in [1/2, 2]\}$. Then, the following $L^\infty - L^\infty$ estimate holds:

$$\big\|\mathcal{F}^{-1}_{\eta\to x}\big(\mathrm{e}^{-i\tau|\eta|}f(\eta)\big)\big\|_{L^\infty(\mathbb{R}^n_x)} \lesssim (1+\tau)^{-\frac{n-1}{2}} \sum_{|\alpha|\leqslant s} \big\|D^\alpha_\eta f(\eta)\big\|_{L^\infty(\mathbb{R}^n_\eta)},$$

where $s > (n+3)/2$.

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